From global class field concepts and modular representations to the conjectures of Shimura-Taniyama-Weil, Birch-Swinnerton-Dyer and Riemann

C. Pierre

Institut de Mathématique pure et appliquée
Université de Louvain
Chemin du Cyclotron, 2
B-1348 Louvain-la-Neuve, Belgium
pierre@math.ucl.ac.be

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Abstract

Based upon new global class field concepts leading to two-dimensional global Langlands correspondences, a modular representation of cusp forms is proposed in terms of global elliptic bisemimodules which are (truncated) Fourier series over $\mathbb{R}$. As application, the conjectures of Shimura-Taniyama-Weil, Birch-Swinnerton-Dyer and Riemann are analyzed.

Introduction

The main objective of this paper consists in providing a modular representation of cusp forms in terms of global elliptic semimodules which are (truncated) Fourier series over $\mathbb{R}$ whose terms correspond to the places of the considered (semi)fields.

This challenge, directly connected to global Langlands correspondences [Lan], depends on the following central result:

At every place of a (CM) (semi)field corresponds a conjugacy class of the Borel subgroup of the general (linear)group. This conjugacy class is associated with a sublattice of Hecke and is in one-to-one correspondence with the corresponding class of representation of the global Weil-Deligne group defined in this paper in such a way that the Hecke eigenvalues and the Frobenius eigenvalues coincide.

More concretely, the symmetric splitting semifields $\widetilde{F}_L$ and $\widetilde{F}_R$ are introduced as symmetric algebraic extensions of a global number field k of characteristic zero. In the complex case, $\widetilde{F}_L$ and $\widetilde{F}_R$ are respectively composed of a set of complex and conjugate complex simple roots of a polynomial ring over k . In the real case, the left and right symmetric semifields are noted respectively $\widetilde{F}_L^+$ and $\widetilde{F}_R^+$.

The infinite places of these semifields are characterized by the algebraic extension degrees of their completions. For example, the n-th place of F_L^+ is defined with respect to the algebraic extension degree

$$[\widetilde{F}^+_{v_n} : k] \simeq f_{v_n} \cdot N = n \cdot N$$

of the pseudo-ramified extension $\widetilde{F}^+_{v_n}$, associated with the completion $F^+_{v_n}$ where $f_{v_n} = n$ is its global class residue degree and where N is the degree of an irreducible algebraic extension and also the order of the global inertia subgroup $I_{F^+_{v_n}}$. This global inertia subgroup can be viewed as the subgroup of inner automorphisms of Galois with respect to the Galois subgroup $\mathrm{Gal}(\widetilde{F}^+_{v_n}/k)$ which will be considered as a subgroup of modular automorphisms of Galois of the algebraic general linear subsemigroup $T_2(\widetilde{F}^+_{v_n})$.

Let $T_2(F_v^+)$ be the algebraic group of upper triangular matrices over the set of completions F_v^+ of F_L^+ at the set of real places $v = \{V_1, \cdots, V_n, \cdots, V_s\}$.

We are interested in the enveloping (semi)algebra $B_R \otimes B_L \simeq T_2^t(F^+_{\overline{v}}) \times T_2(F^+_v)$ of the division (semi)algebra $B_L \simeq T_2(F^+_v)$ because $T_2^t(F^+_{\overline{v}}) \times T_2(F^+_v)$, denoted $\mathrm{GL}_2(F^+_{\overline{v}} \times F^+_v)$, is an algebraic bilinear semigroup "having a representation" in the tensor product $M_{F^+_{\overline{v}}} \otimes M_{F^+_v}$ of a right $T_2^t(F^+_{\overline{v}})$-semimodule $M_{F^+_{\overline{v}}}$ by a left $T_2(F^+_v)$-semimodule $M_{F^+_v}$.

Indeed, we show that a linear algebraic group $\mathrm{GL}_2(F^+)$ over a symmetric field $F^+ = F^+_R \cup F^+_L$, having as representation space a 2^2-dimensional vector space, is covered under the conditions of proposition 1.11 by the bilinear algebraic semigroup $\mathrm{GL}_2(F^+_R \times F^+_L)$.

The $\mathrm{GL}_2(F^+_{\overline{v}} \times F^+_v)$-bisemimodule $M_{F^+_{\overline{v}}} \otimes M_{F^+_v}$ decomposes into the direct sum of $\mathrm{GL}_2(F^+_{\overline{v}_n} \times F^+_{v_n})$-subbisemimodules $M_{F^+_{\overline{v}_{n,m_n}}} \otimes M_{F^+_{v_{n,m_n}}}$, having m_n representatives, according to the decomposition of a Hecke bilattice $\Lambda^{(2)}_{\overline{v}} \otimes \Lambda^{(2)}_v$ into subbilattices $\Lambda^{(2)}_{\overline{v}_{n,m_n}} \otimes \Lambda^{(2)}_{v_{n,m_n}}$ at the biplaces $\overline{v}_n \times v_n$ of $F^+_{\overline{v}} \times F^+_v$. And, the ring of endomorphisms of the $\mathrm{GL}_2(F^+_{\overline{v}} \times F^+_v)$-bisemimodule $M_{F^+_{\overline{v}}} \otimes M_{F^+_v}$, decomposing it into the set of subbisemimodules following the subbilattices, is generated by the products $(T_{q_R} \otimes T_{q_L})$ of Hecke operators T_{q_R} and T_{q_L} for $q \nmid N$ and by the products $(U_{q_R} \otimes U_{q_L})$ of U_{q_R} and U_{q_L} for $q \mid N$: it is noted $T_H(N)_R \otimes T_H(N)_L$.

The coset representative of U_{q_L} (resp. U_{q_R}), referring to the upper (resp. lower) half plane, is chosen to be upper (resp. lower) triangular and given by the integral matrix $\left(\begin{smallmatrix} 1 & b_N \\ 0 & q_N \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 1 & 0 \\ b_N & q_N \end{smallmatrix}\right)$) where the b_N and q_N are integers modulo N .

So, $U_{q_R} \otimes U_{q_L}$ has the coset representative

$$g_2(q_N^2, b_N^2) = \left[\begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix}\right] \begin{pmatrix} 1 & 0 \\ 0 & q_N^2 \end{pmatrix}$$

where

$$D_{q_N^2, b_N^2} = u_2(b_N) \times u_2^t(b_N) = \begin{pmatrix} 1 & b_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_N & 1 \end{pmatrix}$$

is interpreted as the decomposition group element associated with the split Cartan subgroup element $\alpha_{q_N^2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q_N^2 \end{smallmatrix}\right)$.

As a consequence, we have the following proposition:

There is an explicit irreducible semisimple (pseudo-)ramified representation $\rho_{\lambda_\pm}$ *, associated with a weight two cusp form,*

$$\rho_{\lambda_\pm} : \quad \mathrm{Gal}(\widetilde{F}^+_{\overline{v}}/k) \times \mathrm{Gal}(\widetilde{F}^+_v/k) \longrightarrow \mathrm{GL}_2(T_H(N)_R \otimes T_H(N)_L)$$

having eigenvalues:

$$\lambda_\pm(q_N^2, b_N^2) = \frac{(1 + b_N^2 + q_N^2) \pm [(1 + b_N^2 + q_N^2)^2 - 4q_N^2]^{\frac{1}{2}}}{2}$$

verifying:

- $\operatorname{trace} \rho_{\lambda_\pm} = 1 + b_N^2 + q_N^2$;

- $\det \rho_{\lambda_\pm} = \lambda_+(q_N^2, b_N^2) \cdot \lambda_-(q_N^2, b_N^2) = q_N^2$.

Let us introduce the (pseudo-)ramified lattice bisemispace

$$X_{S_{R\times L}} = \mathrm{GL}_2(\widetilde{F}_R \times \widetilde{F}_L) \big/ \mathrm{GL}_2(\mathbb{Z} \big/ N\ \mathbb{Z}\,)^2$$

where

- the general bilinear semigroup $\mathrm{GL}_2(\widetilde{F}_R \times \widetilde{F}_L)$ is taken over the product $\widetilde{F}_R \times \widetilde{F}_L$ of the complex number semifields $\widetilde{F}_R$ and $\widetilde{F}_L$;

- $\mathrm{GL}_2((\mathbb{Z} \big/ N\ \mathbb{Z}\,)^2)$ has entries in squares of integers modulo N .

The toroidal compactification of $X_{S_{R\times L}}$ of the Borel-Serre type can be considered as a toroidal projective isomorphism

$$\gamma^c_{R\times L} : \quad X_{S_{R\times L}} \longrightarrow \overline{X}_{S_{R\times L}} \ ,$$

with

$$\overline{X}_{S_{R\times L}} = \mathrm{GL}_2(F_R^T \times F_L^T) \big/ \mathrm{GL}_2(\mathbb{Z} \big/ N\ \mathbb{Z}\,)^2 \approx \mathrm{GL}_2(F^T_{\overline{\omega}} \times F^T_\omega)$$

where

- F_L^T is the toroidal compactified equivalent of $\widetilde{F}_L$;

- F_ω^T denotes the set of toroidal complex completions.

Its boundary is given by:

$$\partial\overline{X}_{S_{R\times L}} = \mathrm{GL}_2(F_R^{+,T} \times F_L^{+,T}) \big/ \mathrm{GL}_2(\mathbb{Z} \big/ N\ \mathbb{Z}\,)^2 \approx \mathrm{GL}_2(F^{+,T}_{\overline{v}} \times F^{+,T}_v)$$

where

- $F_L^{+,T}$ is the toroidal compactified equivalent of $\widetilde{F}_L^+$;

- $F_v^{+,T}$ denotes the set of toroidal real completions;

in such a way that

1. the number of complex places is equal to the number of real places;
2. the multiplicity of the complex places is equal to one;
3. the complex conjugacy class representatives are covered by the multiples of their real equivalents.

The cosets of $\overline{X}_{S_{R\times L}}$ (resp. $\partial\overline{X}_{S_{R\times L}}$) correspond to the conjugacy classes of $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_{\omega})$ (resp. $\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_{v})$).

The analytic representation of the algebraic bilinear semigroup $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_{\omega})$ is given, by means of a Langlands global correspondence, by the product (of the Fourier development) of a right cusp form of weight $k=2$ restricted to the lower half plane by its classical left analogue restricted to the upper half plane.

Similarly, in the real case, a Langlands global correspondence relates the algebraic bilinear semigroup $\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_{v})$ to its analytic representation given by the product, right by left, of two (truncated) Fourier series over $\mathbb{R}$ called global elliptic A_R (resp. A_L)-semimodules.

These Langlands global correspondences are based upon the representation spaces of the algebraic bilinear semigroups given in terms of their conjugacy class representatives characterized by increasing ranks. The double sets of conjugacy class representatives of the envisaged algebraic bilinear semigroups generate two symmetric towers in such a way that the analytic representation of each conjugacy class representative is given by a term of the Fourier development of the cusp form entering into the cuspidal representation of the considered algebraic bilinear semigroup. This allows to give an algebraic-geometric interpretation to the Fourier series.

Let

$$\phi_L(s_L)=\sum_n\sum_b\phi(s_L)_{n,b}\;q^n/\mathbb{Q}_L\;,\quad n\le\infty\;,\quad q=e^{2\pi ix}\;,\quad x\in\mathbb{R}\;,$$

$$(\text{resp.}\quad \phi_R(s_R)=\sum_n\sum_b\phi(s_R)_{n,b}\;q^{-n}/\mathbb{Q}_R\;)$$

be respectively a left (resp. right) global elliptic A_L (resp. A_R)-semimodule where:

- A_L (resp. A_R) is the ring of sections s_L (resp. s_R) of the semisheaf of rings over the $T_2(F^{+,T}_{v})$-semimodule $M_{F^{+,T}_{v}}$ (resp. $T^t_2(F^{+,T}_{\overline{v}})$)-semimodule $M_{F^{+,T}_{\overline{v}}}$);

- $\phi(s_L)_{n,b} \equiv \lambda(n_N^2, b_N^2)$ such that $\lambda(n_N^2, b_N^2)$ is an eigenvalue of the coset representative $g_2(n_N^2, b_N^2)$ of product of Hecke operators.

Then, the inclusion of $\mathrm{GL}_2(F_{\overline{v}}^{+,T} \times F_v^{+,T})$ into $\mathrm{GL}_2(F_{\overline{\omega}}^T \times F_\omega^T)$ in the sense of the Borel-Serre compactification implies the following results:

Let $S_L(f)$ (resp. $S_R(f)$) be the left [semi-]algebra (resp. right [semi-]algebra) of modular forms

$$\begin{array}{ll} f_L(z) = \sum_n a_{n,L}\ q_L^n & \text{(resp.}\quad f_R(z) = \sum_n a_{n,R}\ q_R^n\ ) \\ q_L = e^{2\pi iz} & (resp.\quad q_R = e^{-2\pi iz}\ ), \quad z \in \mathbb{C}\ , \end{array}$$

- being normalized eigenforms of Hecke operators related to the congruence subgroup $\Gamma_L(N)$ (resp. $\Gamma_R(N)$) introduced in Section 2.12,
- characterized by a weight two and a level N .

On the other hand, let $S_L(\phi)$ (resp. $S_R(\phi)$) denote the left [semi-]algebra (resp. right [semi-]algebra) of global elliptic A_L-semimodules $\phi_L(s_L)$ (resp. A_R-semimodules $\phi_R(s_R)$) in the sense that $f_L(z) \simeq \phi_L(s_L)$(resp. $f_R(z) \simeq \phi_R(s_R)$).

Then, we have the following inclusions of left [semi-]algebras (resp. right [semi-]algebras):

$$\begin{array}{rccc} & S_L(\phi) & \hookrightarrow & S_L(f) \\ \text{(resp.} & S_R(\phi) & \hookrightarrow & S_R(f)\ ). \end{array}$$

And, if the bisemialgebras of modular forms and of global elliptic semimodules are introduced respectively by $S_R(f) \otimes S_L(f)$ and by $S_R(\phi) \otimes S_L(\phi)$, then the inclusion of these bisemialgebras can be stated by:

$$S_R(\phi) \otimes S_R(\phi) \quad \hookrightarrow \quad S_R(f) \otimes S_L(f)\ .$$

Remark that a global elliptic A_L-semimodule $\phi_L(s_L)$ (resp. A_R-semimodule $\phi_R(s_R)$) constitutes an automorphic representation of a modular form $f_L(z)$ (resp. $f_R(z)$) in the sense that the modular representation of $f_L(z)$ (resp. $f_R(z)$) can be given by a set of n , $1 \leq n \leq \infty$, two-dimensional semitori $T_L^2[n]$ (resp. $T_R^2[n]$), restricted to the upper (resp. lower) half plane and covered each one by m_n semicircles of the " n-th class" of the global elliptic semimodule $\phi_L(s_L)$ (resp. $\phi_R(s_R)$).

This kind of modular representation is used to analyze the conjectures of Shimura-Taniyama-Weil, Birch-Swinnerton-Dyer and Riemann.

The hyperbolic uniformization of arithmetic type of an elliptic curve $E(\mathbb{Q})$ is studied by envisaging its modular representation by means of the surjective mapping $\mathcal{H}^{\text{cusp (res)}}_{\mathrm{GL}^{\text{res}}_{2_{\mathbb{R}}}\to E(\mathbb{Q})}$ of the restricted global elliptic A_{R-L}-bisemimodule $\phi_R(s_R)_{\text{res}}\otimes_D\phi_L(s_L)_{\text{res}}$ into $E(\mathbb{Q})$, where $\otimes_D$ denotes a "diagonal" tensor product (see section 2.20).

The modular representation of $E(\mathbb{Q})$ given by

$$\mathcal{H}^{\text{cusp (res)}}_{\mathrm{GL}^{\text{res}}_{2_{\mathbb{R}}}\to E(\mathbb{Q})}: \quad \phi_R(s_R)_{\text{res}}\otimes_D\phi_L(s_L)_{\text{res}}\to E(\mathbb{Q})$$

can be worked out from the p sets of surjective mappings:

$$\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}\to E(\mathbb{F}_p)$$

where $E_f(p_N,m_p)$ is a semicircle of rank $p_N=p\cdot N$ entering into the covering of the semitorus $T^2[p]$, for all prime p taken into account in the restricted eulerian product of $L_R(s_-,E(\mathbb{Q})\otimes_D L_L(s_+,E(\mathbb{Q}))$ in such a way that the orbit space of $\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}$ is associated with the elliptic curve $E(\mathbb{F}_p)$.

This hyperbolic uniformization of arithmetic type of the elliptic curve $E(\mathbb{Q})$ then corresponds to the Shimura-Taniyama-Weil conjecture and is related to the problem of Diophantine equations by means of the Mordell-Weil group of $E(\mathbb{Q})$.

The Birch-Swinnerton-Dyer conjecture is analyzed in the same context. Indeed, it consists in the fact that, if $L_R(s_-,E(\mathbb{Q}))$ and $L_L(s_+,E(\mathbb{Q}))$ are the L-subseries attached to an elliptic curve $E(\mathbb{Q})$ and introduced in chapter 3, then the "pseudo-unramified" rank of $E(\mathbb{Q})$ is the order of vanishing of these L-subseries at $s=1$. The trivial zeros and non-trivial zeros of these L-subseries, defined with respect to the set of p primes envisaged above, are evaluated with respect to the Riemann conjecture which can be treated as follows:

Taking into account that the trivial zeros of the classical zeta function are equal at a sign to the global class residue degrees multiplied by a factor 2 and that a one-to-one correspondence must exist between trivial zeros and pairs of non-trivial zeros, we are led to formulate the proposition: Let $D_{4n^2,i^2}\cdot\varepsilon_{4n^2}$ be a coset representative of the Lie algebra of the decomposition group $D_{i^2}(\mathbb{Z})$ and let α_{4n^2} be the corresponding split Cartan subgroup element. Then, the products of the pairs of the trivial zeros of the Riemann zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$ are mapped into the products of the corresponding pairs of the non-

trivial zeros following:

$$\begin{aligned} D_{4n^2,i^2}\cdot\varepsilon_{4n^2}: \quad \det(\alpha_{4n^2}) &\longrightarrow \det(D_{4n^2,i^2}\cdot\varepsilon_{4n^2}\cdot\alpha_{4n^2})_{ss}\ ,\\ \{(-2n)\cdot(-2n)\} &\longrightarrow \{\lambda_+(4n^2,i^2,E_{4n^2})\cdot\lambda_-(4n^2,i^2,E_{4n^2})\}\ ,\quad \forall\ n\in I\!N\ ,\end{aligned}$$

where $\varepsilon_{4n^2} = \left(\begin{smallmatrix} E_{4n^2} & 0\\ 0 & 1\end{smallmatrix}\right)$ is the infinitesimal generator of the Lie algebra $(\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$.

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1 From global class field concepts to modular representations of general bilinear semigroups

1.1 Preliminaries on semiobjects

The developments of this paper will essentially concern symmetric objects in such a way that the considered mathematical objects can be cut into two symmetric semiobjects $\mathcal{O}_R$ and $\mathcal{O}_L$. The left semiobject $\mathcal{O}_L$ will be localized in (or will refer to) the upper half space while the corresponding right semiobject $\mathcal{O}_R$ will be localized in (or will refer to) the lower half space.

The right semiobject $\mathcal{O}_R$ is then the dual of the left semiobject $\mathcal{O}_L$ and the interest of considering a symmetric object " $\mathcal{O}$ ", decomposed into two dual semiobjects $\mathcal{O}_R$ and $\mathcal{O}_L$, is that the informations concerning the internal mathematical structure of " $\mathcal{O}$ " can be obtained from the product $\mathcal{O}_R\times\mathcal{O}_L$ of the semiobjects $\mathcal{O}_R$ and $\mathcal{O}_L$. Indeed, every endomorphism E of the object " $\mathcal{O}$ " can be decomposed into the product $E_R\times E_L$ of a right endomorphism E_R acting on the right semiobject $\mathcal{O}_R$ by the opposite left endomorphism E_L acting on the corresponding left semiobject $\mathcal{O}_L$ such that $E_R=E_L^{-1}$.

The existence of symmetric objects can be established by the following considerations on function fields.

Let k be a global number field of characteristic zero. Let $k[x_1,\cdots,x_r]$ be a polynomial ring over k and let $R=R_R\cup R_L$ be a symmetric finite extension of k as introduced in the following.

- Let $\boldsymbol{I_L=\{P_\mu(x_1,\cdots,x_r)\mid P_\mu(V_L)=0\}}$ **be the ideal of** $\boldsymbol{k[x_1,\cdots,x_r]}$ in such a way that:

  a) $P_\mu(V_L)$ be the polynomial function in $k[V_L]$ represented by $P_\mu(x_1,\cdots,x_r)$;

b) $V_L \subset R_L$ be an affine semispace restricted to the upper half space.

- Let $\boldsymbol{I_R = \{P_\mu(-x_1,\cdots,-x_r) \mid P_\mu(V_R)=0\}}$ **be the symmetric ideal of** $\boldsymbol{I_L}$ obtained under the involution

$$\tau : \quad P_\mu(x_1,\cdots,x_r) \longrightarrow P_\mu(-x_1,\cdots,-x_r)$$

in such a way that:

a) $P_\mu(V_R)$ be the polynomial function in $k[V_R]$ represented by $P_\mu(-x_1,\cdots,-x_r)$;

b) $V_R \subset R_R$ be an affine semispace restricted to the lower half space, symmetric of V_L and disjoint of V_L or possibly connected to V_L on a symmetry axis plane.

- **The quotient ring** obtained modulo the ideal I_L (resp. I_R) is $Q_L = k[x_1,\cdots,x_r]\big/ I_L$ (resp. $Q_R = k[x_1,\cdots,x_r]\big/ I_R$) [Wat].

  Q_L and Q_R are quotient algebras characterized by the corresponding homomorphisms:

$$\phi_L : \quad Q_L \longrightarrow R_L \ , \qquad \phi_R : \quad Q_R \longrightarrow R_R$$

  where R_L and R_R are commutative (division) semirings localized respectively (or referring to) the upper and lower half spaces (commutative semirings are recalled (or introduced) in section 1.3).

  So **the pair of homomorphisms** $\boldsymbol{\phi_L}$ **and** $\boldsymbol{\phi_R}$ sends the general solution to a pair of symmetric solutions respectively in R_L and in R_R.

- On the other hand, let $\widetilde F = \widetilde F_R \cup \widetilde F_L$ be a symmetric "algebraic" finite extension of k .

  Let $T_r(\widetilde F_L)$ (resp. $T_r^t(\widetilde F_R)$) $\subset \mathrm{GL}_n(\centerdot)$ be the group of matrices of dimension r over $\widetilde F_L$ (resp. $\widetilde F_R$) viewed as an operator sending $\widetilde F_L$ (resp. $\widetilde F_R$) into the affine semispace $T^{(r)}(\widetilde F_L)$ (resp. $T^{(r)}(\widetilde F_R)$) of dimension r :

$$T_r(\centerdot) : \quad \widetilde F_L \longrightarrow T^{(r)}(\widetilde F_L) \qquad (\text{resp.} \quad T_r^t(\centerdot) : \quad \widetilde F_R \longrightarrow T^{(r)}(\widetilde F_R)\ )$$

  in such a way that to the indeterminates $(x_1,\cdots,x_\ell,\cdots,x_r,\cdots,x_{1\ell},\cdots,x_{1r})$ of Q_L (resp. $(-x_1,\cdots,-x_\ell,\cdots,-x_r,\cdots,-x_{1\ell},\cdots,-x_{1r})$ of Q_R), $\forall\ x_{\ell r} = x_\ell \times x_r$,

corresponds the homomorphism:

$$\begin{array}{rll} & \phi'_L : & Q_L \longrightarrow \widetilde{F}_L(x_1 \to e_{11}, \cdots, x_\ell = e_{\ell\ell}, \cdots, x_{r\ell} \to e_{r\ell}) \\ (\text{resp.} & \phi'_R : & Q_R \longrightarrow \widetilde{F}_R(-x_1 \to -e_{11}, \cdots, -x_\ell = -e_{\ell\ell}, \cdots, -x_{r\ell} \to e_{r\ell}\ ) \end{array}$$

where

$$\begin{array}{rl} & T_r(\widetilde{F}_L) = \{e_L = (e_{\ell r}) \in T_r(\widetilde{F}_L) \mid P_{T_\mu}(e_{\ell r}) = 0\} \\ (\text{resp.} & T^t_r(\widetilde{F}_R) = \{e_R = (e_{\ell r}) \in T^t_r(\widetilde{F}_R) \mid P_{T_\mu}(e_{\ell r}) = 0\}\ ) \end{array}$$

with the polynomials $P_{T_\mu}(e_{\ell r}) \in k[x]$ (resp. $P_{T_\mu}(e_{\ell r}) \in k[x]$).

- Finally, let X_L (resp. X_R) be the functor from the quotient ring Q_L (resp. Q_R) to the affine semispace V_L (resp. V_R) in such a way that the diagram:

$$\begin{array}{ccc} Q_L & \xrightarrow{\phi'_L} & \widetilde{F}_L(e_{11}, \cdots, e_{rn}) \\ \Big\downarrow{\scriptstyle X_L} & & \Big\downarrow{\scriptstyle T_r(\cdot)} \\ V_L & \xrightarrow[\sim]{\psi_L} & T^{(r)}\widetilde{F}_L) \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} Q_R & \xrightarrow{\phi'_R} & \widetilde{F}_R(-e_{11}, \cdots, e_{rn}) \\ \Big\downarrow{\scriptstyle X_r} & & \Big\downarrow{\scriptstyle T^t_r(\cdot)} \\ V_R & \xrightarrow[\sim]{\psi_R} & T^{(r)}\widetilde{F}_R) \end{array} \right)$$

  commutes.

  Such functor X_L (resp. X_R) from the k-algebra Q_L (resp. Q_R) to the affine semispace V_L (resp. V_R), homeomorphic to $T^{(r)}(\widetilde{F}_L)$ (resp. $T^{(r)}(\widetilde{F}_R)$), is then representable and is a left (resp. right) affine semigroup scheme over k .

The left and right affine semigroup schemes X_L and X_R are said to be symmetric if every element $a_L \in T^{(r)}(\widetilde{F}_L)$, localized in the upper half space, is symmetric to every element $a_R \in T^{(r)}(\widetilde{F}_R)$, localized in the lower half space, with respect to an axis or a plane...

So, the consideration of a sufficiently large polynomial ring as $k[x_1, \cdots, x_n]$ allows to envisage generally objects as being symmetric and being able to be cut into two symmetric semi-objects introduced in a general way in the following section.

1.2 Semistructures

Having shown as a general rule the existence of symmetric objects, we shall now recall or introduce the semistructures which will be used in the following developments.

The condensed notation R, L means "right, resp. left" (and L, R means "left, resp. right").

- A right (resp. left) **semigroup** $G_{R,L}$ is a nonempty set of right (resp. left) elements, localized in (or referring to) the lower (resp. upper) half space, together with a binary operation on $G_{R,L}$, i.e. a function $G_{R,L} \times G_{R,L} \to G_{R,L}$ or $G_{L,L} \times G_{R,L} \to G_{R,L}$.

- A right (resp. left) **monoid** is a right (resp. left) semigroup $G_{R,L}$ which contains an identity element $a_{R,L} \in G_{R,L}$ such that:
$$a_R \cdot e_R = a_R \quad (\text{resp.} \quad e_L \cdot a_L = a_L \ ), \qquad \forall \ a_{R,L} \in G_{R,L} \ .$$

- A right (resp. left) **semiring** is a nonempty set $R_{R,L}$ together with two binary operations (addition and multiplication) such that:
  a) $(R_{R,L}, +)$ is an abelian right (resp. left) semigroup.
  b) $(a_{R,L} \cdot b_{R,L}) \ c_{R,L} = a_{R,L} \ (b_{R,L} \cdot c_{R,L})$, $\forall \ a_{R,L}, b_{R,L}, c_{R,L} \in R_{R,L}$ (associative multiplication).
  c) $a_{R,L} \ (b_{R,L} + c_{R,L}) = a_{R,L} \ b_{R,L} + a_{R,L} \ c_{R,L}$ and $(a_{R,L} + b_{R,L}) \ c_{R,L} = a_{R,L} \ c_{R,L} + b_{R,L} \ c_{R,L}$ (left and right distribution).

- If $R_{R,L}$ is a commutative semiring with identity $\mathbb{1}_{R_{R,L}}$ and no zero divisors, it will be called a right (resp. left) integral domain.

  Furthermore, if every element of $R_{R,L}$ is a unit (right and left invertible), $R_{R,L}$ is a **division semiring**.

  And, a right (resp. left) **semifield** is a commutative division semiring.

- A right (resp. left) **adele semiring** is the product of the primary completions of the right (resp. left) semifield.

- Let $R_{R,L}$ be a right (resp. left) semiring. A right (resp. left) $\boldsymbol{R_{R,L}}$**-semimodule** is an additive abelian right (resp. left) semigroup $M_{R,L}$ together with a function $M_R \times R_{R,L} \to M_R$ (resp. $R_{L,R} \times M_L \to M_L$) such that:
  a) $(a_R + b_R) \ r_{R,L} = a_R \ r_{R,L} + b_R \ r_{R,L}$ where $a_R \ r_{R,L} = (a \cdot r)_R \in M_R$ (resp. $r_{L,R} \ (a_L + b_L) = r_{L,R} \ a_L + r_{L,R} \ b_L$ where $r_{L,R} \ a_L = (r \cdot a)_L \in M_L$), $\forall \ r_{R,L} \in R_{R,L}$, $a_{R,L}, b_{R,L} \in M_{R,L}$.

b) $a_R\,(r_{R,L}+s_{R,L}) = a_R\,r_{R,L}+a_R\,s_{R,L}$ (resp. $(r_{L,R}+s_{L,R})\,a_L = r_{L,R}\,a_L+s_{L,R}\,a_L$), $\forall\ s_{R,L}\in R_{R,L}$.

c) $(a_R\ s_R)\ r_R = a_R\ (s_R\ r_R)$ (resp. $r_L\ (s_L\ a_L) = (r_L\ s_L)\ a_L$).

If $R_{R,L}$ has an identity element $1\!\!1_{R,L}$ such that $a_R\ 1\!\!1_R = a_R$ (resp. $1\!\!1_L\ a_L = a_L$), $M_{R,L}$ is a right (resp. left) unitary $R_{R,L}$ semimodule.

If $R_{R,L}$ is a right (resp. left) division semiring, then the unitary right (resp. left) $R_{R,L}$-semimodule is a right (resp. left) **vector semispace**.

- If $R_{R,L}$ is a commutative semiring with identity, a $\boldsymbol{R_{R,L}}$**-semialgebra** $\boldsymbol{\mathcal{A}_{R,L}}$ is a semiring $\mathcal{A}_{R,L}$ such that:

  a) $(\mathcal{A}_{R,L},+)$ is a unitary right (resp. left) $R_{R,L}$-semimodule.

  b) $(a_R\,b_R)\,r_{R,L} = a_R\,(b_R\,r_{R,L}) = b_R\,(a_R\,r_{R,L}) = (a\,b\,r)_R\in\mathcal{A}_R$ (resp. $r_{L,R}\,(a_L\,b_L) = (r_{L,R}\ a_L)\ b_L = a_L\ (r_{L,R}\ \ b_L) = (a\ b\ r)_L\in\mathcal{A}_L$).

  If $\mathcal{A}_{R,L}$ is a division semiring, then $\mathcal{A}_{R,L}$ is called a **division semialgebra**.

The generation of the global algebraic extension (semi)fields and of the corresponding completions considered in this paper presents some analogy with the construction of local p-adic number fields which will be recalled in the following section.

1.3 Classical notions about local fields

The field K , which is a finite extension of $\mathbb{Q}_p$, is a p-adic field. Let $\mathcal{O}_K$ denote its ring of integers, $\wp_K$ the unique maximal ideal of $\mathcal{O}_K$, $k(\upsilon_K) = k(\wp_K) = \mathcal{O}_K/\wp_K$ its residue field and $\widetilde{\omega}_K$ a uniformiser in $\mathcal{O}_K$. Let $\upsilon_K : K^*\to\mathbb{Z}$ be the unique valuation so that the absolute value on K is defined by $|\ |_K = |\ |_{\upsilon_K}$ with $|x|_K = (\#k(\upsilon_K))^{-\upsilon_K(x)}$ for $x\in K^*$.

The number of elements in $k(\wp_K)$ is $q=p^f$ where $f_{\upsilon_K} = [k(\upsilon_K):\mathbb{F}_p]$ is the residue degree over $\mathbb{Q}_p$. The ideal $\wp_K\mathcal{O}_K$ of $\mathcal{O}_K$ has the form $\wp_K^{e_{\upsilon_K}} = \tilde{\omega}_K^{e_{\upsilon_K}}$ where e_{υ_K} is the ramification degree of K over $\mathbb{Q}_p$.

Then, we have $[K:\mathbb{Q}_P] = e_{\upsilon_K}\cdot f_{\upsilon_K}$ so that $e_{\upsilon_K} = [K:\mathbb{Q}_P]/f_{\upsilon_K}$.

The maximal unramified extension of K is denoted K^{nr} and its completion is $\widehat{K}^{nr}$. The inertia subgroup I_k is such that:

$$\mathrm{Gal}(K^{ac}/K)/I_K \xrightarrow{\sim} \mathrm{Gal}(K^{nr}/K) \xrightarrow{\sim} \mathrm{Gal}(k(\upsilon_K)^{ac}/k(\upsilon_K))\ .$$

The local Weil group $W_K \subset \text{Gal}(K^{ac}/K)$ is the inverse image of $\text{Frob}^{\mathbb{Z}}_{k(\upsilon_K)} \subset \text{Gal}(k(\upsilon_K)^{ac}/k(\upsilon_K))$ in $\text{Gal}(K^{ac}/K)$.

1.4 Infinite places of a global number field

Let k denote a global number field of characteristic 0 and let $k[x]$ be a polynomial ring composed of a family of pairs of polynomials $\{P_{T_\mu}(x), P_{T_\mu}(-x)\}$, $1 < \mu < \infty$. Then, according to section 1.1, the algebraic extension $\widetilde{F}$ of k , assumed to be generally closed, is the symmetric splitting field $\widetilde{F} = \widetilde{F}_R \cup \widetilde{F}_L$ composed of a right extension semifield $\widetilde{F}_R$ and of a left extension semifield $\widetilde{F}_L$ in one-to-one correspondence. If the algebraic extension field of k is complex, then $\widetilde{F}_L$ (resp. $\widetilde{F}_R$) is composed of the set of complex (resp. conjugate complex) simple roots of the polynomial ring $k[x]$, while, if the algebraic extension field is real, then it will be noted $\widetilde{F}^+ = \widetilde{F}^+_R \cup \widetilde{F}^+_L$ where $\widetilde{F}^+_L$ (resp. $\widetilde{F}^+_R$) is the left (resp. right) algebraic extension semifield composed of the set of positive (resp. symmetric negative) simple roots.

The left and right equivalence classes of the local completions of $\widetilde{F}^{(+)}_L$ and of $\widetilde{F}^{(+)}_R$ are the left and right real (resp. complex) infinite places of $\widetilde{F}^{(+)}_L$ and of $\widetilde{F}^{(+)}_R$: they are equal in number and noted, in the real case:

$$v = \{v_1, \cdots, v_n, \cdots, v_s\} \quad \text{and} \quad \overline{v} = \{\overline{v}_1, \cdots, \overline{v}_n, \cdots, \overline{v}_s\}$$

and, in the complex case:

$$\omega = \{\omega_1, \cdots, \omega_n, \cdots, \omega_s\} \quad \text{and} \quad \overline{\omega} = \{\overline{\omega}_1, \cdots, \overline{\omega}_n, \cdots, \overline{\omega}_s\}$$

With reference to the classical p-adic treatment of local fields, “pseudo-unramified” infinite real places will be characterized algebraically by their global class residue degrees given by:

$$[F^{+,nr}_{v_n} : k] = f_{v_n} = n \quad \text{and} \quad [F^{+,nr}_{\overline{v}_n} : k] = f_{\overline{v}_n} = n \ , \quad n \in I\!N \ ,$$

where $F^{+,nr}_{v_n}$ (resp. $F^{+,nr}_{\overline{v}_n}$), denoting a left (resp. right) pseudo-unramified local completion of $\widetilde{F}^+_L$ (resp. $\widetilde{F}^+_R$), is a left (resp. right) k-semimodule whose dimension is given by the global class residue degree $f_{v_n} = f_{\overline{v}_n} = n$.

The integer n is the order of the archimedean valuation v (resp. $\overline{v}$) defined as a homomorphism of $F^{+,nr}_{v_n}$ (resp. $F^{+,nr}_{\overline{v}_n}$) into the group of positive real numbers in such a way that, for all $x_1 \in F^{+,nr}_{v_1}$ (resp. $\overline{x}_1 \in F^{+,nr}_{\overline{v}_1}$) and $x_n \in F^{+,nr}_{v_n}$ (resp. $\overline{x}_n \in F^{+,nr}_{\overline{v}_n}$), we have:

$$v(n\ x_1) \geq v(x_n) \qquad (\text{resp.} \quad \overline{v}(n\ \overline{x}_1) \geq \overline{v}(\overline{x}_n)\ ) \qquad \text{[Koc]}$$

Similarly, (infinite) complex places will be defined by their global class residue degrees:

$$[F^{nr}_{\omega_n}:k]=f_{\omega_n}=n \quad \text{and} \quad [F^{nr}_{\overline{\omega}_n}:k]=f_{\overline{\omega}_n}=n\ , \quad n\in I\!N\ ,$$

where $F^{nr}_{\omega_n}$ (resp. $F^{nr}_{\overline{\omega}_n}$), denoting a left (resp. right) pseudo-unramified local completion of $\widetilde{F}_L$ (resp. $\widetilde{F}_R$), is a left (resp. right) k-semimodule having as dimension the global class residue degree $f_{\omega_n}=f_{\overline{\omega}_n}=n$.

The corresponding pseudo-ramified completions are assumed to be generated from irreducible k-semimodules $F^+_{v^1_n}$ (resp. $F^+_{\overline{v}^1_n}$) of rank N in the real case, $1\le n\le s$, $N\in\mathbb{N}$, and from "irreducible" k-semimodules $F_{\omega^1_n}$ (resp. $F_{\overline{\omega}^1_n}$) of rank $N\cdot m^{(n)}$ in the complex case, where $m^{(n)}=\sup(m_n+1)$ denotes the multiplicity of the n-th real completion covering its complex equivalent as it will be seen in the following.

So, the ranks of the real pseudo-ramified completions $F^+_{v_n}$ (resp. $F^+_{\overline{v}_n}$) will be given by integers modulo N according to:

$$\begin{aligned}[F^+_{v_n}:k] &= *+n\cdot N \\ &\simeq n\cdot N\end{aligned} \qquad\qquad \left(\text{resp.}\quad \begin{aligned}[F^+_{\overline{v}_n}:k] &= *+n\cdot N \\ &\simeq n\cdot N\ \end{aligned}\right)$$

where $*$ denotes an integer inferior to N .

In the complex case, the ranks of the pseudo-ramified completions F_{ω_n} (resp. $F_{\overline{\omega}_n}$) will also be given by the integers modulo N according to:

$$\begin{aligned}[F_{\omega_n}:k] &= *+n\cdot N\cdot m^{(n)} \\ &\simeq n\cdot N\cdot m^{(n)}\end{aligned} \qquad\qquad \left(\text{resp.}\quad \begin{aligned}[F_{\overline{\omega}_n}:k] &= *+n\cdot N\cdot m^{(n)} \\ &\simeq n\cdot N\cdot m^{(n)}\ \end{aligned}\right)$$

in such a way that each complex completion F_{ω_n} (resp. $F_{\overline{\omega}_n}$) be covered by the set of $m^{(n)}$ real completions $F^+_{v_{n,m_n}}$ (resp. $F^+_{\overline{v}_{n,m_n}}$).

Indeed, as a place is an equivalence class of completions, we have to consider at each real place v_n (resp. $\overline{v}_n$) a set of real completions $\{F^+_{v_{n,m_n}}\}_{m_n}$, $m_n\in\mathbb{N}$ (resp. $\{F^+_{\overline{v}_{n,m_n}}\}_{m_n}$), equivalent to $F^+_{v_n}$ (resp. $F^+_{\overline{v}_n}$) (with $m_n=0$), and characterized by the same rank as $F^+_{v_n}$ (resp. $F^+_{\overline{v}_n}$).

Similarly, at each complex place ω_n (resp. $\overline{\omega}_n$), a set of complex completions $\{F_{\omega_{n,m_{\omega_n}}}\}_{m_{\omega_n}}$, $m_{\omega_n}\in\mathbb{N}$ (resp. $\{F_{\overline{\omega}_{n,m_{\omega_n}}}\}_{m_{\omega_n}}$), equivalent to F_{ω_n} (resp. $F_{\overline{\omega}_n}$) and characterized by the same rank as F_{ω_n} (resp. $F_{\overline{\omega}_n}$) has to be considered.

Let $F_\omega=\{F_{\omega_1},\cdots,F_{\omega_{n,m_{\omega_n}}},\cdots,F_{\omega_{s,n_s}}\}$ (resp. $F_{\overline{\omega}}=\{F_{\overline{\omega}_1},\cdots,F_{\overline{\omega}_{n,m_{\overline{\omega}_n}}},\cdots,F_{\omega_{s,n_s}}\}$) denote the set of complex pseudo-ramified completions of the number semifield $\widetilde{F}_L$ (resp. $\widetilde{F}_R$) at the set of complex places ω (resp. $\overline{\omega}$) and let $F^+_v=\{F^+_{v_1},\cdots,F^+_{v_{n,m_n}},\cdots,F^+_{v_{s,m_s}}\}$

(resp. $F^+_{\overline{v}} = \{F^+_{\overline{v}_1}, \cdots, F^+_{\overline{v}_{n,m_n}}, \cdots, F^+_{\overline{v}_{s,m_s}}\}$) be the corresponding real pseudo-ramified completions of the number semifield $\widetilde{F}^+_L$ (resp. $\widetilde{F}^+_L$) at the set of real places v (resp. $\overline{v}$).

Then, the direct sum of the complex pseudo-ramified completions is given by:

$$F_{\omega_\oplus} = \bigoplus_n \bigoplus_{m_{\omega_n}} F_{\omega_{n,m_{\omega_n}}} \qquad (\text{resp.} \quad F_{\overline{\omega}_\oplus} = \bigoplus_n \bigoplus_{m_{\omega_n}} F_{\overline{\omega}_{n,m_{\omega_n}}} \ )$$

while the direct sum of the real pseudo-ramified completions is given by:

$$F^+_{v_\oplus} = \bigoplus_n \bigoplus_{m_n} F^+_{v_{n,m_n}} \qquad (\text{resp.} \quad F^+_{\overline{v}_\oplus} = \bigoplus_n \bigoplus_{m_n} F^+_{\overline{v}_{n,m_n}} \ )$$

On the other hand, if the global class residue degree is a prime p or an integer modulo p , then $F^{+,nr}_{v_p}$ and $F^{+,nr}_{v_{p+i}}$ can refer respectively to a completion of an (unramified) p-adic (semi)field and to a completion of an extension of this p-adic (semi)field which can be identified to a finite extension of $\mathbb{Q}_p$; indeed, $F^{+,nr}_{v_p}$ and $F^{+,nr}_{v_{p+i}}$ are pseudo-unramified local (semi)fields at p elements which correspond to p Galois automorphisms. More concretely, $F^{+,nr}_{v_p}$ and $F^{+,nr}_{v_{p+i}}$ are characterized by their global class residue degrees given by:

$$[F^{+,nr}_{v_p} : k] = f_{v_p} = p \quad \text{and} \quad [F^{+,nr}_{v_{p+i}} : k] = f_{v_{p+i}} = k\ p + i' = p + i \ ,$$

$$k \in I\!N \,, \quad 0 \leq i' \leq p-1 \,, \quad 1 \leq i \leq \infty \ .$$

Remark that $F^{+,nr}_{v_p}$ and $F^{+,nr}_{v_{p+i}}$ are (unramified) p-adic fields if:

1. $f_{v_p} = p$ and $f_{v_{p+i}} = kp + i' = p^\alpha$, $\alpha \in \mathbb{N}$, respectively.
2. the number of nonunits of $F^{+,nr}_{v_p}$ and of $F^{+,nr}_{v_{p+i}}$ is a power of p .
3. $F^{+,nr}_{v_p}$ and $F^{+,nr}_{v_{p+i}}$ are considered as completions of $\mathbb{Q}$ in the p-adic metric.

In the pseudo-ramified case, we should have:

$$[F^+_{v_{p+i}} : k] \simeq f_{v_{p+i}} \cdot N = (k\ p + i') \cdot N = (p+i) \cdot N$$

where $F^+_{v_{p+i}}$ is a local (semi)field at p elements corresponding to $(p+i) \cdot N$ Galois automorphisms from a global point of view. $F^+_{v_{p+i}}$ is a p-adic field if:

1. $f_{v_{p+i}} \cdot N \cdot \#(Nu) = p^\beta$, $\beta \in \mathbb{N}$, where $\#(Nu)$ is the number of nonunits of $F^+_{v_{p+i}}$.
2. $F^+_{v_{p+i}}$ is defined as a completion of $\mathbb{Q}$ with respect to the p-adic metric.

If the number of left and right real places of the completions of $\widetilde{F}_L^+$ and $\widetilde{F}_R^+$ are equal respectively to the number of left and right complex places of the completions of $\widetilde{F}_L$ and of $\widetilde{F}_R$, then $\widetilde{F}=\widetilde{F}_R\cup\widetilde{F}_L$ can be interpreted as a CM number field given by $\widetilde{F}=\widetilde{F}^+\cdot\widetilde{E}$ where $\widetilde{F}^+$ is the real number field and where $\widetilde{E}$ is an imaginary quadratic field.

Then a finite extension K of $\mathbb{Q}_p$ can be identified to the completion of $\widetilde{F}^+$ in a place v_1' above p . And, as p is decomposed in E into two places $\wp$ and $\overline{\wp}$, the places of $\widetilde{F}$ dividing $\wp$ are divided into a set of left complex places $\omega'=\{\omega_1',\cdots,\omega_n',\cdots,\omega_s'\}$ above $\wp$ belonging to the completions of $\widetilde{F}_L$ and into a set of right conjugate places $\overline{\omega}'=\{\overline{\omega}_1',\cdots,\overline{\omega}_n',\cdots,\overline{\omega}_s'\}$ above $\overline{\wp}$ belonging to the completions of $\widetilde{F}_R$.

1.5 Galois subgroups and global inertia subgroups

Let $\mathrm{Gal}(\widetilde{F}_{v_n}^{+,nr}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+,nr}/k)$) be the Galois subgroup of the pseudo-unramified extension $\widetilde{F}_{v_n}^{+,nr}$ (resp. $\widetilde{F}_{\overline{v}_n}^{+,nr}$) of k corresponding to the pseudo-unramified completion $F_{v_n}^{+,nr}$ (resp. $F_{\overline{v}_n}^{+,nr}$), and let $\mathrm{Gal}(\widetilde{F}_{v_n}^{+}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+}/k)$) be the Galois subgroup of the pseudo-ramified extension $\widetilde{F}_{v_n}^{+}$ (resp. $\widetilde{F}_{\overline{v}_n}^{+}$) of k corresponding to the pseudo-ramified completion $F_{v_n}^{+}$ (resp. $F_{\overline{v}_n}^{+}$).

Then, the global inertia subgroup $I_{F_{v_n}^+}$ (resp. $I_{F_{\overline{v}_n}^+}$) of $\mathrm{Gal}(\widetilde{F}_{v_n}^{+}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+}/k)$), being the Galois subgroup of the irreducible extension $\widetilde{F}_{v_n^1}^{+}$ (resp. $F_{\overline{v}_n^1}^{+}$), can be defined by:

$$\begin{aligned} I_{F_{v_n}^+} &= \mathrm{Gal}(\widetilde{F}_{v_n}^{+}/k)\Big/\mathrm{Gal}(\widetilde{F}_{v_n}^{+,nr}/k) \\ (\text{resp.}\quad I_{F_{\overline{v}_n}^+} &= \mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+}/k)\Big/\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+,nr}/k)\ ) \end{aligned}$$

leading to the exact sequence:

$$\begin{array}{cccccccccc} & 1 & \longrightarrow & I_{F_{v_n}^+} & \longrightarrow & \mathrm{Gal}(\widetilde{F}_{v_n}^{+}/k) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_{v_n}^{+,nr}/k) & \longrightarrow & 1 \\ (\text{resp.} & 1 & \longrightarrow & I_{F_{\overline{v}_n}^+} & \longrightarrow & \mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+}/k) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+,nr}/k) & \longrightarrow & 1\ ). \end{array}$$

The global inertia subgroup $I_{F_{v_n}^+}$ (resp. $I_{F_{\overline{v}_n}^+}$) of order N can then be viewed as the normal subgroup or inner automorphisms of Galois with respect to the Galois subgroup $\mathrm{Gal}(\widetilde{F}_{v_n}^{+}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{v}_n}^{+}/k)$) which can be considered as a subgroup of modular automorphisms of Galois [Pie1].

Similar conclusions are obtained for the Galois subgroups $\mathrm{Gal}(\widetilde{F}_{\omega_n}/k)$ (resp. $\mathrm{Gal}(\widetilde{F}_{\overline{\omega}_n}/k)$) of the pseudo-ramified extension $\widetilde{F}_{\omega_n}$ (resp.$\widetilde{F}_{\overline{\omega}_n}$) of k .

As we are concerned with Galois class fields, all the "(pseudo-)ramification"" orders are supposed to be equal to the same positive integer N in the real case and to the positive integers $N \cdot m^{(n)}$ in the complex case.

1.6 Global Weil semigroups

Let $\widetilde F^+_{v_n,m_n}$ (resp. $\widetilde F^+_{\overline v_n,m_n}$) denote a real pseudo-ramified extension referring to the infinite place v_n (resp. $\overline v_n$) and let $\widetilde F^{+,nr}_{v_n,m_n}$ (resp. $\widetilde F^{+,nr}_{\overline v_n,m_n}$) denote the corresponding pseudo-unramified extension.

Then, we set:

$$\mathrm{Gal}(\widetilde F^+_v/k) = \bigoplus_{n,m_n} \mathrm{Gal}(\widetilde F^+_{v_n,m_n}/k)$$

$$(\text{resp.}\quad \mathrm{Gal}(\widetilde F^+_{\overline v}/k) = \bigoplus_{n,m_n} \mathrm{Gal}(\widetilde F^+_{\overline v_n,m_n}/k)\ )$$

as well as

$$\mathrm{Gal}(\widetilde F^{+,nr}_v/k) = \bigoplus_{n,m_n} \mathrm{Gal}(\widetilde F^{+,nr}_{v_n,m_n}/k)$$

$$(\text{resp.}\quad \mathrm{Gal}(\widetilde F^{+,nr}_{\overline v}/k) = \bigoplus_{n,m_n} \mathrm{Gal}(\widetilde F^{+,nr}_{\overline v_n,m_n}/k)\ ).$$

As in the p-adic case, the Weil group is the Galois subgroup of the elements inducing on the residue field an integer power of a Frobenius element, we shall assume that, in the characteristic zero case, the Weil group will be the Galois subgroup of the pseudo-ramified extensions characterized by extension degrees $d = 0 \bmod N$.

In this respect, let $\dot{\widetilde F}{}^+_{v_n}$ (resp. $\dot{\widetilde F}{}^+_{\overline v_n}$) denote a pseudo-ramified Galois extension characterized by the degree

$$[\dot{\widetilde F}{}^+_{v_n} : k] \equiv [\dot{\widetilde F}{}^+_{\overline v_n} : k] = n \cdot N$$

in such a way that the sum of the closed global Weil sub(semi)groups is given by:

$$\mathrm{W}_{F^+_v} = \bigoplus_{n,m_n} \mathrm{Gal}(\dot{\widetilde F}{}^+_{v_{n,m_n}}/k)$$

$$(\text{resp.}\quad \mathrm{W}_{F^+_{\overline v}} = \bigoplus_{n,m_n} \mathrm{Gal}(\dot{\widetilde F}{}^+_{\overline v_{n,m_n}}/k)\ ).$$

And, the product of $W_{F^+_{\overline v}}$ and of $W_{F^+_v}$ gives:

$$\mathrm{W}_{F^+_{\overline v}} \times \mathrm{W}_{F^+_v} = \bigoplus_{n,m_n} \left(\mathrm{Gal}(\dot{\widetilde F}{}^+_{\overline v_{n,m_n}}/k) \times \mathrm{Gal}(\dot{\widetilde F}{}^+_{v_{n,m_n}}/k) \right) .$$

1.7 Representations of algebraic bilinear semigroups

Let $B_L/\widetilde{F}_v^+$ denote a left division semialgebra of dimension r over the semifield $\widetilde{F}_v^+ = \{\widetilde{F}_{v_1}^+, \cdots, \widetilde{F}_{v_{s,m_s}}^+\}$ and let $B_R/\widetilde{F}_{\overline{v}}^+$ denote the corresponding right division semialgebra of the same dimension r over the semifield $\widetilde{F}_{\overline{v}}^+ = \{\widetilde{F}_{\overline{v}_1}^+, \cdots, \widetilde{F}_{\overline{v}_{s,m_s}}^+\}$ in such a way that:

- B_R is the opposite semialgebra of B_L ;

- the center of B_L is in one-to-one correspondence with the center of B_R .

If it is assumed that the division semialgebra B_L is isomorphic to the matrix algebra $T_r(\widetilde{F}_v^+)$ of Borel upper triangular matrices over $\widetilde{F}_v^+$ and that the opposite division semialgebra B_R is isomorphic to the matrix algebra $T_r^t(\widetilde{F}_{\overline{v}}^+)$ of Borel lower triangular matrices over $\widetilde{F}_{\overline{v}}^+$, then we have:

$$B_R \otimes B_L \simeq T_r^t(\widetilde{F}_{\overline{v}}^+) \times T_r(\widetilde{F}_v^+) \equiv \mathrm{GL}_r(\widetilde{F}_{\overline{v}}^+ \times \widetilde{F}_v^+) \ .$$

So, the tensor product $B_R \otimes B_L$ of the division semialgebras B_R and B_L is isomorphic to the product of the group $T_r^t(\widetilde{F}_{\overline{v}}^+)$ of lower triangular matrices by the group $T_r(\widetilde{F}_v^+)$ of upper triangular matrices. Such a product $T_r^t(\widetilde{F}_{\overline{v}}^+) \times T_r(\widetilde{F}_v^+)$ is denoted $\mathrm{GL}_r(\widetilde{F}_{\overline{v}}^+ \times \widetilde{F}_v^+)$ and corresponds to a general bilinear semigroup having a representation in the tensor product $M_R \otimes M_L$ of a right $T_r^t(\widetilde{F}_{\overline{v}}^+)$-semimodule M_R by a left $T_r(\widetilde{F}_v^+)$-semimodule M_L .

So, the general bilinear semigroup

$$\mathrm{GL}_r(\widetilde{F}_{\overline{v}}^+ \times \widetilde{F}_v^+) = T_r^t(\widetilde{F}_{\overline{v}}^+) \times T_r(\widetilde{F}_v^+) \ ,$$

has a representation in a $\mathrm{GL}_r(\widetilde{F}_{\overline{v}}^+ \times \widetilde{F}_v^+)$-bisemimodule $M_R \otimes M_L$ of dimension r^2 .

1.8 Proposition

Let $B_L/\widetilde{F}_v^+$ (resp. $B_R/\widetilde{F}_{\overline{v}}^+$) be a left (resp. right) division semialgebra of dimension r over the extension semifield $\widetilde{F}_v^+$ (resp. $\widetilde{F}_{\overline{v}}^+$) such that

$$B_L \simeq T_r(\widetilde{F}_v^+) \ , \qquad B_R \simeq T_r^t(\widetilde{F}_{\overline{v}}^+) \ .$$

Then, we have

$$B_R \otimes B_L \simeq \mathrm{GL}_r(\widetilde{F}_{\overline{v}}^+ \times \widetilde{F}_v^+)$$

implying that the general bilinear semigroup $\mathrm{GL}_r(\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v)$ *has the natural bilinear Gauss decomposition:*

$$\begin{aligned} g_r(\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v) &= t^t_r(\widetilde{F}^+_{\overline{v}}) \times t_r(\widetilde{F}^+_v) \\ &= [u^t_r(\widetilde{F}^+_{\overline{v}}) \times u_r(\widetilde{F}^+_v)] \times [d_r(\widetilde{F}^+_{\overline{v}}) \times d_r(\widetilde{F}^+_v)] \end{aligned}$$

for any regular matrix

- $g_r(\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v) \in \mathrm{GL}_r(\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v)$ *;*

- $u_r(\widetilde{F}^+_{\overline{v}}) \in UT_r(\widetilde{F}^+_v)$ *where* $UT_r(\widetilde{F}^+_v)$ *is the group of upper unitriangular matrices;*

- $d_r(\widetilde{F}^+_v) \in D_r(\widetilde{F}^+_v)$ *where* $D_r(\widetilde{F}^+_v)$ *is the group of diagonal matrices.*

Proof. The classical case

$$B^e \simeq \mathrm{End}_{\widetilde{F}^+}(B) \simeq \mathrm{End}_{\widetilde{F}^+}((\widetilde{F}^+)^r) \simeq \mathcal{M}_r(\widetilde{F}^+)$$

where:

- $B^e = B \times_F B^{\mathrm{op}}$ with B a central simple F-algebra and with B^{op} the opposite algebra,

- $\mathrm{End}_{\widetilde{F}^+}(B)$ is isomorphic to B^e if B is an Azumaya algebra (then, B is a projective F-module),

- $\widetilde{F}^+$ is a ring in such a way that $\widetilde{F}^+ = \widetilde{F}^+_{\overline{v}} \cup \widetilde{F}^+_v$,

- $\mathcal{M}_r(\widetilde{F}^+)$ is the ring of $r \times r$ matrices with entries in $\widetilde{F}^+$,

becomes in the considered bilinear case:

$$B^e \equiv B_R \otimes B_L \simeq \mathrm{End}_{\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v}(B_R \times B_L) \simeq \mathrm{End}_{\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v}((\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v)^r) \simeq \mathrm{GL}_r(\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v)$$

where:

- B_L (resp. B_R) is a (division) central simple $\widetilde{F}^+_v$ (resp. $\widetilde{F}^+_{\overline{v}}$)-semialgebra of dimension r over $\widetilde{F}^+_v$ (resp. $\widetilde{F}^+_{\overline{v}}$);

- $\mathrm{End}_{\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v}(B_R \times B_L) \simeq \mathrm{End}_{\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v}(B_L) \simeq \mathrm{End}_{\widetilde{F}^+_{\overline{v}} \times \widetilde{F}^+_v}(B_R)$. ∎

1.9 Proposition

Let $F^+ = F_R^+ \cup F_L^+ \simeq \widetilde{F}_{\overline{v}}^+ \cup \widetilde{F}_v^+$ be a compact real algebraically closed field defined from the symmetric algebraically closed semifields $F_R^+ \simeq \widetilde{F}_{\overline{v}}^+$ and $F_L^+ \simeq \widetilde{F}_v^+$.

Let $\mathrm{GL}_r(F^+)$ be the group of invertible $(r \times r)$ matrices with entries in F^+ . The algebraic general linear group $\mathrm{GL}_r(F^+)$ has for representation space a vectorial space V of dimension r^2 isomorphic to $(F^+)^{r^2}$.

On the other hand, let $\mathrm{GL}_r(F_R^+ \times F_L^+)$ be the algebraic bilinear semigroup, introduced in sections 1.7 and 1.8, and having for representation space the $\mathrm{GL}_r(F_R^+ \times F_L^+)$-bisemimodule $M_R \otimes M_L$ of dimension r^2 .

Then, we can state the one-to-one correspondence:

$$\begin{aligned} I_{\mathrm{GL}} : \quad \mathrm{GL}_r(F^+) &\xrightarrow{\ \sim\ } \mathrm{GL}_r(F_R^+ \times F_L^+) \; , \\ V &\xrightarrow{\ \sim\ } M_R \otimes M_L \; , \end{aligned}$$

between the algebraic linear group $\mathrm{GL}_r(F^+)$ and the algebraic bilinear semigroup $\mathrm{GL}_r(F_R^+ \times F_L^+)$ sending the vectorial space V into the $\mathrm{GL}_r(F_R^+ \times F_L^+)$-bisemimodule $M_R \otimes M_L$.

Proof. M_L (resp. M_R) is a r-dimensional vector semispace over F_L^+ (resp. F_R^+). Every basis of M_L (resp. M_R) defines an isomorphism of the group $T_r(M_L)$ (resp. $T_r^t(M_R)$) of the automorphisms of M_L (resp. M_R) with $T_r(F_L^+)$ (resp. $T_r^t(F_R^+)$) $\subset \mathrm{GL}_r(F_R^+ \times F_L^+)$.
And, every basis of $(M_R \otimes M_L)$ defines an isomorphism of the group $\mathrm{GL}_r(M_R \otimes M_L)$ of the automorphisms of $(M_R \otimes M_L)$ with $\mathrm{GL}_r(F_R^+ \times F_L^+)$.
Now, $\mathrm{GL}_r(F_R^+ \times F_L^+)$ has the bilinear Gauss decomposition:

$$g_r(F_R^+ \times F_L^+) = (u_r^t(F_R^+) \times u_r(F_L^+)) \times (d_r(F_R^+) \times d_r(F_L^+))$$

for any matrix $g_r(F_R^+ \times F_L^+) \in \mathrm{GL}_r(F_R^+ \times F_L^+)$,
while $\mathrm{GL}_R(F^+)$ has the Gauss decomposition:

$$g_r(F^+) = (u_r^t(F^+) \times u_r(F^+)) \times d_r(F^+)$$

for $g_r(F^+) \in \mathrm{GL}_r(F^+)$.
If we take into account the maps $(\star)$

$$\begin{cases} u_r^t(F^+) & \longrightarrow u_r^t(F_R^+) \; , \\ u_r(F^+) & \longrightarrow u_r(F_L^+) \; , \\ d_r(F^+) & \longrightarrow d_r(F_R^+ \times F_L^+) \; , \end{cases} \tag{$\star$}$$

then,

1. $V \simeq M_R \otimes M_L$ in such a way that the basis of the n^2-dimensional vector space V corresponds to the basis of the n^2-dimensional $\mathrm{GL}_r(F_R^+ \times F_L^+)$-bisemimodule $M_R \otimes M_L$;

2. the linear algebraic group $\mathrm{GL}_r(F^+)$ is covered by the bilinear algebraic semigroup $\mathrm{GL}_r(F_R^+ \times F_L^+)$. ∎

1.10 Proposition

Let $F^+ = F_R^+ \cup F_L^+$ be a symmetric algebraic extension field.

If the Gauss decompositions of the linear algebraic group $\mathrm{GL}_r(F^+)$ and of the bilinear algebraic semigroup $\mathrm{GL}_r(F_R^+ \times F_L^+)$ correspond under the conditions $(\star)$ of proposition 1.9, then:

1. $\mathrm{GL}_r(F^+) \xrightarrow{\sim} \mathrm{GL}_r(F_R^+ \times F_L^+)$ *expressing that* $\mathrm{GL}_r(F^+)$ *is covered by* $\mathrm{GL}_r(F_R^+ \times F_L^+)$;

2. *the r^2-dimensional representation space $V = \mathrm{Rep\ sp}(\mathrm{GL}_r(F^+))$ of $\mathrm{GL}_r(F^+)$ coincides with the r^2-dimensional representation space $M_R \otimes M_L = \mathrm{Rep\ sp}(\mathrm{GL}_r(F_R^+ \times F_L^+))$ of $\mathrm{GL}_r(F_R^+ \times F_L^+)$.*

We are then led to a bilinear version of the Wedderburn theorem [K-M-R-T]:

1.11 Proposition

Let B_L and B_R be the division semialgebras respectively over the semifields $F_L^+ \simeq \widetilde F_v^+$ and $F_R^+ \simeq \widetilde F_{\overline v}^+$. The following conditions are then equivalent:

a) *$B_R \otimes B_L$ is central simple of dimension r^2 .*

b) $B_R \otimes B_L \simeq \mathrm{GL}_r(F_R^+ \times F_L^+)$,

   $B_L \otimes B_R \simeq \mathrm{GL}_r(F_L^+ \times F_R^+)$.

c) *The canonical map* $\begin{cases} B_R \otimes B_L \to \mathrm{End}_{F_R^+ \times F_L^+}(B_L) \ , \\ B_L \otimes B_R \to \mathrm{End}_{F_L^+ \times F_R^+}(B_R) \ , \end{cases}$ *which relates respectively to* $\begin{cases} b_R \otimes b_L \\ b_L \otimes b_R \end{cases}$ *the map* $\begin{cases} x^2 \to b_R x^2 b_L \ , \\ x^2 \to b_L x^2 b_R \ , \end{cases}$ *is an isomorphism.*

d) $\begin{cases} B_R/F_R^+ \simeq T_r^t(F_R^+) \\ B_L/F_L^+ \simeq T_r(F_L^+) \end{cases}$ *implies* $\quad B_R/F_R^+ \otimes B_L/F_L^+ \simeq T_r^t(F_R^+) \times T_r(F_L^+)$.

1.12 Proposition

1. *Every involution of the first kind on the central simple F_L^+-semialgebra B_L (resp. F_R^+-semialgebra B_R) is defined by a mapping $\sigma_{L\to R} : B_L/F_L^+ \to B_R/F_R^+$ (resp. $\sigma_{R\to L} : B_R/F_R^+ \to B_L/F_L^+$) such that: $t_r(F_L^+) \to t_r^t(F_R^+)$ (resp. $t_r^t(F_R^+) \to t_r(F_L^+)$) transposes every upper (resp. lower) Borel upper triangular matrix $t_r(F_L^+) \in T_r(F_L^+)$ (resp. $t_r^t(F_R^+) \in T_r^t(F_R^+)$).*

2. *Every involution of the second kind $\sigma_{R\times L\to L\times R}$ on the tensor product of the central simple F_R^+-semialgebra B_R by the central simple F_L^+-semialgebra B_L is given by the exchange involution:*

$$\sigma_{R\times L\to L\times R} : \quad B_R/F_R^+ \otimes B_L/F_L^+ \longrightarrow B_L/F_L^+ \otimes B_R/F_R^+$$

such that

$$t_r^t(F_R^+) \times t_r(F_L^+) \longrightarrow t_r(F_L^+) \times t_r^t(F_R^+)$$

resulting in a double involution of the first kind:

$$\sigma_{R\times L\to L\times R} = \sigma_{R\to L} \circ \sigma_{L\to R} \ .$$

Proof. This is a direct consequence of Propositions 1.9 and 1.12. However, let us prove it more explicitly inspired by the book of [K-M-R-T].

Let $\gamma : M_R \times M_L \to F_R^+ \times F_L^+$ be a bilinear form on the product of the semifields F_R^+ and F_L^+ and let

$$\begin{aligned} \widehat{\gamma} : &\quad M_L \longrightarrow M_R = \mathrm{Hom}(M_L, F_R^+) \ , \\ \widehat{\gamma}^{-1} : &\quad M_R \longrightarrow M_L = \mathrm{Hom}(M_R, F_L^+) \ . \end{aligned}$$

For any $f_L \in \mathrm{End}_{F_L^+}(M_L)$ (resp. $f_R \in \mathrm{End}_{F_R^+}(M_R)$), we define $\sigma_\gamma(f_L) \in \mathrm{End}_{F_L^+}(M_L)$ by $\sigma_\gamma(f_L) = \widehat{\gamma}^{-1} \circ f_R \circ \widehat{\gamma}$ (resp. $\sigma_{\gamma^{-1}}(f_R) \in \mathrm{End}_{F_R^+}(M_R)$ by $\sigma_{\gamma^{-1}}(f_R) = \widehat{\gamma} \circ f_L \circ \widehat{\gamma}^{-1}$). It is then clear that any involution of this first kind $\sigma_{L\to R}$ (resp. $\sigma_{R\to L}$) on a division semialgebra B_L (resp. B_R) can be explained as follows:

$$\begin{aligned} (B_L, \sigma_{L\to R}) &\simeq \mathrm{End}_{B_L}(M_L) \simeq (T_r(F_L^+), \sigma_\gamma) \ , \\ (B_R, \sigma_{R\to L}) &\simeq \mathrm{End}_{B_R}(M_R) \simeq (T_r^t(F_R^+), \sigma_{\gamma^{-1}}) \ . \end{aligned}$$ ∎

2 Modular representations of general bilinear semigroups in terms of global elliptic bisemimodules

Until the end of this paper, all developments will refer to the real and complex dimensions which are inferior or equal to 2.

Let us summarize our terminology:

- We consider the left and right symmetric completions F_ω and $F_{\overline{\omega}}$ decomposing respectively according to the set of places $\omega = \{\omega_1, \cdots, \omega_n, \cdots, \omega_s\}$ and $\overline{\omega} = \{\overline{\omega}_1, \cdots, \overline{\omega}_n, \cdots, \overline{\omega}_s\}$) of the complex algebraic extension semifields $\widetilde{F}_L$ and $\widetilde{F}_R$ of a global number field of characteristic zero k . If the algebraic extension semifields $\widetilde{F}_L^+$ and $\widetilde{F}_R^+$ are real, then the corresponding left and right symmetric completions F_v^+ and $F_{\overline{v}}^+$ decompose according to the set of real places $v = \{v_1, \cdots, v_n, \cdots, v_s\}$ and $\overline{v} = \{\overline{v}, \cdots, \overline{v}_n, \cdots, \overline{v}_s\}$.

- In the real case, let $B_{F_v^+}$ (resp. $B_{F_{\overline{v}}^+}$)be a left (resp. right) division semialgebra over the completions F_v^+ (resp. $F_{\overline{v}}^+$). Assuming the isomorphisms:

  $$B_{F_v^+} \simeq T_2(F_v^+) \qquad \text{and} \qquad B_{F_{\overline{v}}^+} \simeq T_2^t(F_{\overline{v}}^+) \ ,$$

  we have

  $$B_{F_{\overline{v}}^+} \otimes B_{F_v^+} \simeq T_2^t(F_{\overline{v}}^+) \times T_2(F_v^+) \equiv \mathrm{GL}_2(F_{\overline{v}}^+ \times F_v^+)$$

  where $\mathrm{GL}_2(F_{\overline{v}}^+ \times F_v^+)$ is a general bilinear semigroup, with entries in the product of the semifield completions $F_{\overline{v}}^+ \times F_v^+$, generated from the product, right by left, of triangular matrix (semi)groups of order 2 which leads to a bilinear Gauss decomposition as introduced in proposition 1.9.

- $GL_2(F_{\overline{v}}^+ \times F_v^+)$ has a representation in the $B_{F_{\overline{v}}^+} \otimes B_{F_v^+}$-bisemimodule $M_{F_{\overline{v}}^+} \otimes M_{F_v^+}$ composed of the set $\{M_{F_{\overline{v}_{n,m_n}}^+} \otimes M_{F_{v_{n,m_n}}^+}\}_{n=1,m_n}^s$ of subbisemimodules $M_{F_{\overline{v}_{n,m_n}}^+} \otimes M_{F_{v_{n,m_n}}^+}$ following the places $\overline{v}_n$ and v_n considered with their m_n representatives.

- Similarly, if $F_{v_\oplus}^+ = \bigoplus\limits_{n,m_n} F_{v_{n,m_n}}^+$ (resp. $F_{\overline{v}_\oplus}^+ = \bigoplus\limits_{n,m_n} F_{\overline{v}_{n,m_n}}^+$) denotes the sum of the real pseudo-ramified completions, the algebraic bilinear semigroup $\mathrm{GL}_2(F_{\overline{v}_\oplus}^+ \times F_{v_\oplus}^+)$, with entries in the product of $F_{\overline{v}_\oplus}^+$ by $F_{v_\oplus}^+$, has a representation in the $B_{F_{\overline{v}_\oplus}^+} \otimes B_{F_{v_\oplus}^+}$-

bisemimodule $M_{F^+_{\overline{v}_\oplus}}\otimes M_{F^+_{v_\oplus}}$ which decomposes according to:

$$M_{F^+_{\overline{v}_\oplus}}\otimes M_{F^+_{v_\oplus}}=\underset{n,m_n}{\oplus}\left(M_{F^+_{\overline{v}_{n,m_n}}}\otimes M_{F^+_{v_{n,m_n}}}\right)$$

in such a way that:

$$B_{F^+_{\overline{v}_\oplus}}\otimes B_{F^+_{v_\oplus}}\simeq T^t_2(F^+_{\overline{v}_\oplus})\times T_2(F^+_{v_\oplus})\equiv \mathrm{GL}_2(F^+_{\overline{v}_\oplus}\times F^+_{v_\oplus})\ .$$

- The representation space $(M_{F^+_{\overline{v}}}\otimes M_{F^+_v})$ of $\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v)$ is homomorphic to the representation space $(M_{F^+_{\overline{v}_\oplus}}\otimes M_{F^+_{v_\oplus}})$ of $\mathrm{GL}_2(F^+_{\overline{v}_\oplus}\times F^+_{v_\oplus})$:

$$\begin{array}{ccc}
\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v) & \longrightarrow & \mathrm{GL}_2(F^+_{\overline{v}_\oplus}\times F^+_{v_\oplus})\\
\Big\downarrow & & \Big\downarrow\\
(M_{F^+_{\overline{v}}}\otimes F^+_v) & \longrightarrow & (M_{F^+_{\overline{v}_\oplus}}\otimes F^+_{v_\oplus})
\end{array}$$

2.1 Definition: Left and right real (pseudo-)ramified lattices

In this respect, a left (resp. right) maximal order $\Lambda^{(2)}_v$ (resp. $\Lambda^{(2)}_{\overline{v}}$) over $\mathbb{Z}/N\,\mathbb{Z}$ in the left (resp. right) division semialgebra $B_{F^+_v}$ (resp. $B_{F^+_{\overline{v}}}$) can be introduced by:

a) the isomorphisms

$$\Lambda^{(2)}_v\simeq T_2(\mathbb{Z}/N\,\mathbb{Z})\qquad\text{and}\qquad \Lambda^{(2)}_{\overline{v}}\simeq T^t_2(\mathbb{Z}/N\,\mathbb{Z})$$

where $T_2(\mathbb{Z}/N\,\mathbb{Z})$ is the subgroup of upper triangular matrices over the integers modulo N as it will be developed in section 2.3.

b) its representation in a left (resp. right) lattice decomposing according to the set of places v (resp. $\overline{v}$) of F^+_v (resp. $F^+_{\overline{v}}$) following:

$$\Lambda^{(2)}_{v_\oplus}=\underset{n,m_n}{\oplus}\Lambda^{(2)}_{v_{n,m_n}}\qquad(\text{resp.}\quad \Lambda^{(2)}_{\overline{v}_\oplus}=\underset{n,m_n}{\oplus}\Lambda^{(2)}_{\overline{v}_{n,m_n}}\ )$$

where $\Lambda^{(2)}_{v_{n,m_n}}$ (resp. $\Lambda^{(2)}_{\overline{v}_{n,m_n}}$) is a left (resp. right) (Hecke) pseudo-ramified sublattice associated with the n-th place v_n (m_n-th representative) and corresponding to the $T_2(F^+_{v_n})$-subsemimodule $M_{F^+_{v_{n,m_n}}}$ (resp. $T^t_2(F^+_{\overline{v}_n})$-subsemimodule $M_{F^+_{\overline{v}_{n,m_n}}}$).

2.2 Real (pseudo-)ramified bilattices

The tensor product $\Lambda^{(2)}_{\overline{v}} \otimes \Lambda^{(2)}_{v}$ of the right (pseudo-) ramified lattice $\Lambda^{(2)}_{\overline{v}}$ by the left (pseudo-)ramified lattice $\Lambda^{(2)}_{v}$ constitutes the representation space of the (bilinear) arithmetic group $\mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2 \equiv T^t_2(\mathbb{Z}/N\,\mathbb{Z}) \times T_2(\mathbb{Z}/N\,\mathbb{Z})$. Indeed we have that:

$$\mathrm{Rep\ sp}(\mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2) = \Lambda^{(2)}_{\overline{v}} \otimes \Lambda^{(2)}_{v}$$

is homomorphic to $\Lambda^{(2)}_{\overline{v}_\oplus} \otimes \Lambda^{(2)}_{v_\oplus}$ which decomposes into the direct sum of bisublattices $\Lambda^{(2)}_{\overline{v}_{n,m_n}} \otimes \Lambda^{(2)}_{v_{n,m_n}}$, according to the set of biplaces $\overline{v} \times v$ of $F^+_{\overline{v}} \times F^+_{v}$:

$$\Lambda^{(2)}_{\overline{v}_\oplus} \otimes \Lambda^{(2)}_{v_\oplus} = \bigoplus_n \bigoplus_{m_n} (\Lambda^{(2)}_{\overline{v}_{n,m_n}} \otimes \Lambda^{(2)}_{v_{n,m_n}}) \ .$$

On the other hand, the (bilinear) arithmetic group $\mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$ has the following bilinear Gauss decomposition:

$$g_2(\mathbb{Z}/N\,\mathbb{Z})^2 = [d_2(\mathbb{Z}/N\,\mathbb{Z}) \times d_2(\mathbb{Z}/N\,\mathbb{Z})] \times [u^t_2(\mathbb{Z}/N\,\mathbb{Z}) \times u_2(\mathbb{Z}/N\,\mathbb{Z})]$$

for every element $g_2(\mathbb{Z}/N\,\mathbb{Z})^2 \in \mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$ where

- $d_2(\mathbb{Z}/N\,\mathbb{Z})$ is a diagonal matrix of order 2, also called a split Cartan subgroup element;

- $u_2(\mathbb{Z}/N\,\mathbb{Z})$ is a two-dimensional upper unitriangular matrix;

- $u^t_2(\mathbb{Z}/N\,\mathbb{Z})$ is a two-dimensional lower unitriangular matrix.

2.3 Complex (pseudo-)ramified bilattices

The complex case can be handled similarly as it was done for the real case.

a) So, let B_{F_ω} (resp. $B_{F_{\overline{\omega}}}$) be a left (resp. right) division semialgebra over the complex completions F_ω (resp. $F_{\overline{\omega}}$).

   Setting $B_{F_\omega} \simeq T_2(F_\omega)$ and $B_{F_{\overline{\omega}}} \simeq T_2(F_{\overline{\omega}})$, the complex bilinear semigroup $\mathrm{GL}_2(F_{\overline{\omega}} \times F_\omega)$ with entries in the product $F_{\overline{\omega}} \times F_\omega$ of the complex semifield completions can be introduced by:

$$B_{F_{\overline{\omega}}} \otimes B_{F_\omega} \simeq T^t_2(F_{\overline{\omega}}) \times T_2(F_\omega) \equiv \mathrm{GL}_2(F_{\overline{\omega}} \times F_\omega) \ .$$

The $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$-bisemimodule $M_{F_{\overline{\omega}}}\otimes M_{F_\omega}$ is homomorphic to $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$ which decomposes into:

$$M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}=\bigoplus_{n,m_{\omega_n}}(M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F_{\omega_{n,m_{\omega_n}}}})\ ,$$

following the complex places $\omega_n\in\omega$ (resp. $\overline{\omega}_n\in\overline{\omega}$) with representatives m_{ω_n} , constitutes a representation of $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$.

b) This can be justified if a left (resp. right) lattice $\Lambda^{(2)}_\omega$ (resp. $\Lambda^{(2)}_{\overline{\omega}}$) over $\mathbb{Z}/N\ \mathbb{Z}$ in the left (resp. right) division semialgebra B_{F_ω} (resp. $B_{F_{\overline{\omega}}}$) is introduced by the isomorphisms:

$$\Lambda^{(2)}_\omega\simeq T_2(\mathbb{Z}/N\ \mathbb{Z})\qquad(\text{resp.}\quad\Lambda^{(2)}_{\overline{\omega}}\simeq T^t_2(\mathbb{Z}/N\ \mathbb{Z})\ )$$

where $T_2(\mathbb{Z}/N\ \mathbb{Z})$ is the subgroup of upper triangular matrices over the integers modulo N according to section 1.4.

This left (resp. right) lattice $\Lambda^{(2)}_\omega$ (resp. $\Lambda^{(2)}_{\overline{\omega}}$) is homomorphic to $\Lambda^{(2)}_{\omega_\oplus}$ (resp. $\Lambda^{(2)}_{\overline{\omega}_\oplus}$) which decomposes following the set of complex places ω (resp. $\overline{\omega}$) according to:

$$\Lambda^{(2)}_{\omega_\oplus}=\bigoplus_{n,m_{\omega_n}}\Lambda^{(2)}_{\omega_{n,m_{\omega_n}}}\qquad(\text{resp.}\quad\Lambda^{(2)}_{\overline{\omega}_\oplus}=\bigoplus_{n,m_{\omega_n}}\Lambda^{(2)}_{\overline{\omega}_{n,m_{\omega_n}}}\ ).$$

c) The tensor product $\Lambda^{(2)}_{\overline{\omega}}\otimes\Lambda^{(2)}_\omega$ constitutes the representation space of $\mathrm{GL}_2(\mathbb{Z}/N\ \mathbb{Z})^2\equiv T^t_2(\mathbb{Z}/N\ \mathbb{Z})\times T_2(\mathbb{Z}/N\ \mathbb{Z})$ in such a way that: $\Lambda^{(2)}_{\overline{\omega}_\oplus}\otimes\Lambda^{(2)}_{\omega_\oplus}$ decomposes into the bisublattices according to:

$$\Lambda^{(2)}_{\overline{\omega}_\oplus}\otimes\Lambda^{(2)}_{\omega_\oplus}=\bigoplus_{n,m_{\omega_n}}(\Lambda^{(2)}_{\overline{\omega}_{n,m_{\omega_n}}}\otimes\Lambda^{(2)}_{\omega_{n,m_{\omega_n}}}).$$

2.4 Borel-Serre compactification type of the (pseudo-)ramified lattice bisemispace

Let us introduce the (pseudo-)ramified complex lattice bisemispace $X_{S_{R\times L}}$ as the quotient semigroup:

$$X_{S_{R\times L}}=\mathrm{GL}_2(\widetilde{F}_R\times\widetilde{F}_L)/\mathrm{GL}_2(\mathbb{Z}/N\ \mathbb{Z})^2\approx M_{F_{\overline{\omega}}}\otimes M_{F_\omega}$$

where the complex algebraic bilinear semigroup $\mathrm{GL}_2(\widetilde{F}_R\times\widetilde{F}_L)$ is taken over the product $\widetilde{F}_R\times\widetilde{F}_L$ of the symmetric (algebraically closed) splitting semifields $\widetilde{F}_R$ and $\widetilde{F}_L$.

$X_{S_{R\times L}}$ has a representation in a $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$-bisemimodule $M_{F_{\overline{\omega}}}\otimes M_{F_\omega}$ in the sense that:

$$X_{S_{R\times L}}\approx M_{F_{\overline{\omega}}}\otimes M_{F_\omega}\ .$$

The Borel-Serre toroidal compactification of $X_{S_{R\times L}}$ can be considered as the toroidal projective isomorphism of compactification given by:

$$\gamma^c_{R\times L}: \quad X_{S_{R\times L}} \longrightarrow \overline{X}_{S_{R\times L}}$$

where

$$\overline{X}_{S_{R\times L}} = \mathrm{GL}_2(F^T_R \times F^T_L) \big/ \mathrm{GL}_2(\mathbb{Z}/N\ \mathbb{Z})^2 \approx M_{F^T_{\overline{\omega}}} \otimes M_{F^T_\omega}$$

such that:

- $X_{S_{R\times L}}$ may be viewed as the interior of $\overline{X}_{S_{R\times L}}$ in the sense that $\gamma^c_{R\times L}$ is an inclusion isomorphism: $\quad X_{S_{R\times L}} \hookrightarrow \overline{X}_{S_{R\times L}}$;

- F^T_R and F^T_L are "toroidal" symmetric algebraic(ally closed) semifields;

- $M_{F^T_{\overline{\omega}}} \otimes M_{F^T_\omega}$ is the toroidal equivalent of $M_{F_{\overline{\omega}}} \otimes M_{F_\omega}$.

2.5 Proposition

If the multiplicity of the complex places is equal to one, the boundary $\partial\overline{X}_{S_{R\times L}}$ of $\overline{X}_{S_{R\times L}}$ is introduced as resulting from the surjective morphism:

$$\gamma^\delta_{R\times L}: \quad \overline{X}_{S_{R\times L}} \to \partial\overline{X}_{S_{R\times L}}$$

sending the complex toroidal (pseudo-)ramified lattice bisemispace $\overline{X}_{S_{R\times L}}$ into its boundary $\partial\overline{X}_{S_{R\times L}}$ given by

$$\partial\overline{X}_{S_{R\times L}} = \mathrm{GL}_2(F^{+,T}_R \times F^{+,T}_L) \big/ \mathrm{GL}_2(\mathbb{Z}/N\ \mathbb{Z})^2 \approx M_{F^{+,T}_{\overline{v}}} \otimes M_{F^{+,T}_v} ,$$

where

- *$F^{+,T}_R$ and $F^{+,T}_L$ are "toroidal" symmetric algebraic(ally closed) semifields;*

- *$M_{F^{+,T}_{\overline{v}}} \otimes M_{F^{+,T}_v}$ is the toroidal equivalent of $M_{F^+_{\overline{v}}} \otimes M_{F^+_v}$;*

- *$F^{+,T}_v$ (resp. $F^{+,T}_{\overline{v}}$) are the toroidal completions associated with F^+_v (resp. $F^+_{\overline{v}}$) at the set of real places v (resp. $\overline{v}$)* [Pie2].

Proof.

a) According to section 2.3, the complex toroidal $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$-bisemimodule $M_{F^T_{\overline{\omega}}}\otimes M_{F^T_\omega}$ is the representation space of $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^+_\omega)$ in such a way that:

$$\text{Rep sp}(\mathrm{GL}_2(F^T_{\overline{\omega}_\oplus}\times F^T_{\omega_\oplus}))=M_{F^T_{\overline{\omega}_\oplus}}\otimes M_{F^T_{\omega_\oplus}}=\bigoplus_{n,m_{\omega_n}}(M_{F^T_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F^T_{\omega_{n,m_{\omega_n}}}})$$

where $F^T_{\overline{\omega}}$ and F^T_ω denote the sets of toroidal completions corresponding respectively to $F_{\overline{\omega}}$ and F_ω .

Similarly, the real $B_{F^{+,T}_{\overline{v}}}\otimes B_{F^{+,T}_v}$-bisemimodule $M_{F^{+,T}_{\overline{v}}}\otimes M_{F^{+,T}_v}$ is the representation space of $\mathrm{GL}_2(F^{;T}_{\overline{v}}\times F^{+,+}_v)$ in such a way that:

$$\text{Rep sp}(\mathrm{GL}_2(F^{+,T}_{\overline{v}_\oplus}\times F^{+,T}_{v_\oplus}))=M_{F^{+,T}_{\overline{v}_\oplus}}\otimes M_{F^{+,T}_{v_\oplus}}=\bigoplus_{n,m_n}(M_{F^{;T}_{\overline{v}_{n,m_n}}}\otimes M_{F^{+,T}_{v_{n,m_n}}})\ .$$

b) At the condition that $m^{(\omega_n)}=1$, $\forall\ n$, the surjective morphism: $\gamma^\delta_{R\times L}:\overline{X}_{S_{R\times L}}\to\partial\overline{X}_{S_{R\times L}}$, defining the boundary of the Borel-Serre compactification, corresponds to a covering of Rep sp$(\mathrm{GL}_2^{(\mathrm{res})}(F^T_{\overline{\omega}}\times F^T_\omega))$ by Rep sp$(\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v))$ in such a way that:

1) the representation space of the “restricted” algebraic bilinear semigroup $\mathrm{GL}_2^{(\mathrm{res})}(F^T_{\overline{\omega}}\times F^T_\omega)$ be given by:

$$\text{Rep sp}(\mathrm{GL}_2^{(\mathrm{res})}(F^T_{\overline{\omega}}\times F^T_\omega))=M^{\mathrm{res}}_{F^T_{\overline{\omega}}}\otimes M^{\mathrm{res}}_{F^T_\omega}$$

homomorphic to $M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}\otimes M^{\mathrm{res}}_{F^T_{\omega_\oplus}}=\bigoplus\limits_n(M_{F^T_{\overline{\omega}_n}}\otimes M_{F^T_{\omega_n}})$;

2) each complex subbisemimodule

$$(M_{F^T_{\overline{\omega}_n}}\otimes M_{F^T_{\omega_n}})\ ,\qquad 1\le n\le s\le\infty\ ,\quad m_{\omega_n}=1\ ,$$

be covered by the $m^{(n)}$ real subbisemimodules $\{(M_{F^{+,T}_{\overline{v}_{n,m_n}}}\otimes M_{F^{+,T}_{v_{n,m_n}}})\}^{m^{(n)}}_{m_n=1}$.

c) So, the complex “bi”points of the boundary of $\overline{X}_{S_{R\times L}}$ are in one-to-one correspondence with the real “bi”points of $\partial\overline{X}_{S_{R\times L}}$. ∎

2.6 Corollary

The surjective morphism

$$\gamma^\delta_{R,L}:\quad \overline{X}_{S_{R,L}}\to\partial\overline{X}_{S_{R,L}}$$

is a morphism of inclusion in the sense that:

a) the real lattice $\Lambda_v^{(2)}$ (resp. $\Lambda_{\overline{v}}^{(2)}$) is included into the corresponding restricted complex lattice $\Lambda_\omega^{(2)(\mathrm{res})} = \sum\limits_n \Lambda_{\omega_n}^{(2)}$ (resp. $\Lambda_{\overline{\omega}}^{(2)(\mathrm{res})} = \sum\limits_n \Lambda_{\overline{\omega}_n}^{(2)}$):

$$\Lambda_v^{(2)} \subset \Lambda_\omega^{(2)(\mathrm{res})} \qquad (\textit{resp.} \quad \Lambda_{\overline{v}}^{(2)} \subset \Lambda_{\overline{\omega}}^{(2)(\mathrm{res})} \ ).$$

b) the real lattice $\Lambda_v^{(2)} = \bigoplus\limits_{n,m_n} \Lambda_{v_{n,m_n}}^{(2)}$ is commensurable with the restricted complex lattice $\Lambda_\omega^{(2)(\mathrm{res})} = \sum\limits_n \Lambda_{\omega_n}^{(2)}$ if and only if:

$$\Lambda_{\omega_n}^{(2)} = \bigoplus_{m_n} \Lambda_{v_{n,m_n}}^{(2)} \ , \quad 1 \le n \le s \ .$$

Proof. In correspondence with the covering of the representation space $M_{F_{\overline{\omega}}^T}^{\mathrm{res}} \otimes M_{F_\omega^T}^{\mathrm{res}}$ of the restricted algebraic bilinear semigroup $\mathrm{GL}_2^{(\mathrm{res})}(F_{\overline{\omega}}^T \times F_\omega^T)$ by the representation space $M_{F_{\overline{v}}^{+,T}} \otimes M_{F_v^{+,T}}$ of the real bilinear semigroup $\mathrm{GL}_2(F_{\overline{v}}^{+,T} \times F_v^{+,T})$ as given in proposition 2.5, we have that $\Lambda_{\overline{\omega}}^{(2)(\mathrm{res})} \otimes \Lambda_\omega^{(2)(\mathrm{res})}$ is covered by $\Lambda_{\overline{v}}^{(2)} \otimes \Lambda_v^{(2)}$.
It then results that $\Lambda_{\omega_n}^{(2)} = \bigoplus\limits_{m_n} \Lambda_{v_{n,m_n}}^{(2)}$ (resp. $\Lambda_{\overline{\omega}_n}^{(2)} = \bigoplus\limits_{m_n} \Lambda_{\overline{v}_{n,m_n}}^{(2)}$). ■

2.7 Bilinear parabolic subsemigroup

Let $P_2(F_{[\omega_1]}^T)$ (resp. $P_2(F_{[\overline{\omega}_1]}^T)$) be a minimal parabolic (locally compact) subgroup over irreducible toroidal completions of F_ω (resp. $F_{\overline{\omega}}$), i.e. restricted to a global class residue degree $f_{\omega_1} = 1$ (resp. $f_{\overline{\omega}_1} = 1$). Let then

$$P_2(F_{[\overline{\omega}_1]}^T) \times F_{[\omega_1]}^T) \equiv P_2(F_{[\overline{\omega}_1]}^T) \times P_2(F_{[\omega_1]}^T)$$

denote a bilinear complex parabolic semigroup: it is the smallest (pseudo-)ramified normal bilinear subsemigroup of $\mathrm{GL}_2(F_{\overline{\omega}}^T \times F_\omega^T)$ constituting a representation of the product $I_{F_{\overline{\omega}_n}} \times I_{F_{\omega_n}}$, $1 \le n \le s$, of the global inertia subgroups $I_{F_{\overline{\omega}_n}}$ and $I_{F_{\omega_n}}$. Remark that $\mathrm{GL}_2(F_{\overline{\omega}}^T \times F_\omega^T)$ acts by conjugation on $P_2(F_{[\overline{\omega}_1]}^T \times F_{[\omega_1]}^T)$. Then, a double coset decomposition of $\mathrm{GL}_2(F_R^T \times F_L^T)$ leads to the following compactified bisemispace:

$$\overline{S}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}^{P_2} = P_2(F_{[\overline{\omega}_1]}^T \times F_{[\omega_1]}^T) \setminus \mathrm{GL}_2(F_R^T \times F_L^T) / \mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$$

such that the modular conjugacy classes of $\mathrm{GL}_2(F_{\overline{\omega}}^T \times F_\omega^T)$ (with respect to $(I_{F_{\overline{\omega}_n}} \times I_{F_{\omega_n}})$) correspond to the cosets of $P_2(F_{[\overline{\omega}_1]}^T \times F_{[\omega_1]}^T) \setminus \mathrm{GL}_2(F_R^T \times F_L^T)$.

The toroidal isomorphism of compactification $\gamma_{R\times L}^c : X_{S_{R\times L}} \to \overline{X}_{S_{R\times L}}$ with boundary $\partial\overline{X}_{S_{R\times L}}$ leads to define the boundary $\partial\overline{S}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}^{P_2}$ of $\overline{S}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}^{P_2}$ by:

$$\partial\overline{S}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}^{P_2} = P_2(F_{[\overline{v}_1]}^{+,T} \times F_{[v_1]}^{+,T}) \setminus \mathrm{GL}_2(F_R^{+,T} \times F_L^{+,T}) / \mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$$

where $P_2(F^{+,T}_{[\overline{v}_1]}\times F^{+,T}_{[v_1]})$ is the bilinear real parabolic semigroup over the product $F^{+,T}_{[\overline{v}_1]}\times F^{+,T}_{[v_1]}$ of irreducible toroidal completions of $F^+_{\overline{v}}\times F^+_v$.
$\partial\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$ is the equivalent of a Shimura (bisemi)variety following [Pie2].

It was shown in [Pie1] that the representation space of $(P_2(F^{+,T}_{[v_1]}))$ (resp. $(P_2(F^{+,T}_{[\overline{v}_1]}))$) in a $P_2(F^{+,T}_{[v_1]})$-subsemimodule $M^I_{[v_1]}$ (resp. $P_2(F^{+,T}_{[\overline{v}_1]})$-subsemimodule $M^I_{[\overline{v}_1]}$) is a left (resp. right) quantum of quantum field theory.

2.8 Analytic development of the $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$-bisemimodule $M_{F^T_{\overline{\omega}}}\otimes M_{F^T_\omega}$

It appears from section 2.3 that the complex bilinear algebraic semigroup $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$ has a representation space given by the $B_{F_{\overline{\omega}}}\otimes B_{F_\omega}$-bisemimodule $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$ which decomposes according to the equivalent representatives of the places of $F_{\overline{\omega}}$ and F_ω as follows:

$$M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}=\bigoplus_n\bigoplus_{m_{\omega_n}}(M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F_{\omega_{m,m_m}}})$$

where the set $\{M_{F_{\omega_{n,m_{\omega_n}}}}\}_{F_{\omega_{n,m_{\omega_n}}}}$ (resp. $\{M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}\}_{F_{\overline{\omega}_{n,m_{\omega_n}}}}$) of subsemimodules forms a tower of conjugacy class representatives of $T_2(F_\omega)\subset\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$ (resp. $T^t_2(F_{\overline{\omega}})\subset\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$) characterized by increasing ranks, which are increasing integers modulo N as developed in section 1.5.

Remark also that the decomposition of $(M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}})$ into subbisemimodules also results from the action of the product of Hecke operators $(T_{q_R}\otimes T_{q_L})$ as it will be seen in the following.

So, the $B_{F_{\overline{\omega}}}\otimes B_{F_\omega}$-bisemimodule $M_{F_{\overline{\omega}}}\otimes M_{F_\omega}$ decomposes into a double symmetric tower of conjugacy class representatives corresponding each other respectively in the upper and in the lower half space.

The toroidal projective isomorphism of compactification sends the $B_{F_{\overline{\omega}}}\otimes B_{F_\omega}$-bisemimodule $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$ into the $B_{F^T_{\overline{\omega}}}\otimes B_{F^T_\omega}$-bisemimodule $M_{F^T_{\overline{\omega}_\oplus}}\otimes M_{F^T_{\omega_\oplus}}$ which decomposes following the " s " places of $F^T_{\overline{\omega}}$ and F^T_ω according to:

$$M_{F^T_{\overline{\omega}_\oplus}}\otimes M_{F^T_{\omega_\oplus}}=\bigoplus_{n=1}^{s}\bigoplus_{m_{\omega_n}}\left(M_{F^T_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F^T_{\omega_{n,m_{\omega_n}}}}\right) .$$

With respect to proposition 2.5, we take into account the restricted $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$-bisemimodule

$$M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}\otimes M^{\mathrm{res}}_{F^T_{\omega_\oplus}}=\bigoplus_{n=1}^{s}\left(M_{F^T_{\overline{\omega}_n}}\otimes M_{F^T_{\omega_n}}\right)\ ,\quad m^{(\omega_n)}=1\ ,$$

where the left (resp. right) subsemimodule $M_{F^T_{\omega_n}}$ (resp. $M_{F^T_{\overline{\omega}_n}}$) is a left (resp. right) two-dimensional semitorus $T^2_L[n]$ (resp. $T^2_R[n]$) localized in the upper (resp. lower) half plane.

Indeed, let $\vec{z} = \sum\limits_{\alpha=1}^{2} z_\alpha \ \vec{e}_\alpha$ be a vector of $\mathbb{C}^{\,1}$ and, more precisely, of F^T_ω , and fix $z = \sum\limits_{\alpha=1}^{2} z_\alpha \ |\vec{e}_\alpha|$. Then, every left (resp. right) 2-dimensional real semitorus has the analytic development:

$$T^2_L[n] \simeq \lambda(2,n) \ e^{2\pi inz}$$

$$(\text{resp.} \quad T^2_R[n] \simeq \lambda(2,n) \ e^{-2\pi inz} \ ),$$

where

- $\lambda(2,n)$, introduced in section 2.12, can be considered as a Hecke character;
- $\lambda(2,n) \ e^{2\pi inz}$ corresponds to the product of two (semi)circles localized in perpendicular planes [Pie2].

More concretely, we have that the (semi)torus

$$T^2_L[n] \approx S^1_{d_1}[n] \times S^1_{d_2}[n]$$

is diffeomorphic to the product of two circles $S^1_{d_1}[n]$ and $S^1_{d_2}[n]$ localized in perpendicular planes.

This gives rise to the following decomposition:

$$\begin{aligned} T^2_L[n] &\approx \lambda(2,n) \ e^{2\pi inz} \ , \quad z = x_{d_1} + i \ y_{d_2} \ , \quad x_{d_1} \in \mathbb{R} \ , \quad y_{d_2} \in \mathbb{R} \ , \\ &\simeq S^1_{d_1}[n] \times S^1_{d_2}[n] \\ &= r_{S^1_{d_1}} \ e^{2\pi inx_{d_1}} \times r_{S^1_{d_2}} \ e^{2\pi in(iy_{d_2})} \ , \end{aligned}$$

where $r_{S^1_{d_1}}$ and $r_{S^1_{d_2}}$ are radii verifying

$$\lambda(2,n) \simeq r_{S^1_{d_1}} \times r_{S^1_{d_2}} \ ,$$

in such a way that $e^{2\pi in(iy_{d_2})}$, localized in a plane perpendicular to $e^{2\pi inx_{d_1}}$ and defined over $i\mathbb{R}$, is effectively the equation of a circle.

This is justified by the fact that a rotation of 90^0 of the circle $S^1_{d_2}[n]$ over $i\mathbb{R}$ transforms it into the circle $S^1_{d_{2_\perp}}[n]$ over $\mathbb{R}$ localized in the same plane as the circle $S^1_{d_1}[n]$ according to:

$$\begin{aligned}
\mathrm{rot}(90^0): \quad & S^1_{d_2}[n] \longrightarrow S^1_{d_{2_\perp}}[n] \\
r_{S^1_{d_2}}\ e^{2\pi in(iy_{d_2})} \Big/ i\mathbb{R} & \longrightarrow r_{S^1_{d_2}}\ e^{2\pi in(y_{d_2})} \Big/ \mathbb{R}\ ,
\end{aligned}$$

where:

- $S^1_{d_2}[n] = r_{S^1_{d_2}}[\cos(2\pi iny_{d_2}] + i\sin(2\pi iny_{d_2})]$,

- $S^1_{d_{2_\perp}}[n] = r_{S^1_{d_2}}[\cos(2\pi ny_{d_2}] + i\sin(2\pi ny_{d_2})]$.

Then, $M^{\mathrm{res}}_{F^T_{\omega_\oplus}}$ (resp. $M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}$) has the analytic development:

$$\mathrm{EIS}_L(2,n) \simeq \bigoplus_{n=1}^{s} \lambda(2,n)\ e^{2\pi inz}\ , \quad s \leq \infty\ ,$$

$$(\text{resp.} \quad \mathrm{EIS}_R(2,n) \simeq \bigoplus_{n=1}^{s} \lambda(2,n)\ e^{-2\pi inz}\ , \quad s \leq \infty\ )$$

where $\mathrm{EIS}_L(2,n)$ (resp. $\mathrm{EIS}_R(2,n)$) is the Fourier development of the equivalent of a normalized left (resp. right) Eisenstein series of weight $k=2$ restricted to the upper (resp. lower) half plane [Pie2] and verifying $\lambda(2,n) = \sigma^{\mathrm{res}}_{k-1}(n) \approx n\cdot N$ according to proposition 2.15 while classically $\sigma_{k-1}(n) = \sum\limits_{d|n} d^{k-1}$ [Ser2].

So, $\mathrm{EIS}_L(2,n)$ (resp. $\mathrm{EIS}_R(2,n)$) corresponds to a left (resp. right) generalized cusp form of weight $k=2$ as it will be introduced in the following sections; it can be decomposed into a tower of semitori $T^2_L[n]$ (resp. $T^2_R[n]$), characterized by increasing ranks and representing analytically the conjugacy class representatives of $T_2(F^T_\omega)$ (resp. $T^t_2(F^T_{\overline{\omega}})$) $\subset \mathrm{GL}_2(F^T_{\overline{\omega}} \times F^T_\omega)$).

2.9 Analytic development of the $\mathbf{GL_2}(F^{+,T}_{\overline{v}} \times F^{+,T}_v)$-bisemimodule $M_{F^{+,T}_{\overline{v}}} \otimes M_{F^{+,T}_v}$

According to proposition 2.5, the boundary $\partial\overline{X}_{S_{R,L}}$ of $\overline{X}_{S_{R,L}}$ is given by $\partial\overline{X}_{S_{R,L}} = \mathrm{GL}_2(F^{+,T}_R \times F^{+,T}_L) \big/ \mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$. So, the toroidal compactification of the $B_{F^+_{\overline{v}}} \otimes B_{F^+_v}$-bisemimodule $M_{F^+_{\overline{v}}} \otimes M_{F^+_v}$ is given by the $B_{F^{+,T}_{\overline{v}}} \otimes B_{F^{+,T}_v}$-bisemimodule $M_{F^{+,T}_{\overline{v}}} \otimes B_{F^{+,T}_v}$ constituting the representation space of the real algebraic bilinear semigroup $\mathrm{GL}_2(F^{+,T}_{\overline{v}} \times F^{+,T}_v)$.

$M_{F^{+,T}_{\overline{v}_\oplus}} \otimes M_{F^{+,T}_{v_\oplus}}$ can be developed as a direct sum of subbisemimodules following the " s " places of $F_v^{+,T}$ and $F_{\overline{v}}^{+,T}$ according to:

$$M_{F^{+,T}_{\overline{v}_\oplus}} \otimes M_{F^{+,T}_{v_\oplus}} = \mathop{\oplus}_{n=1}^{s} \mathop{\oplus}_{m_n} (M_{F^{+,T}_{\overline{v}_{n,m_n}}} \otimes M_{F^{+,T}_{v_{n,m_n}}})$$

where $M_{F^{+,T}_{v_{n,m_n}}}$ (resp. $M_{F^{+,T}_{\overline{v}_{n,m_n}}}$) is a left (resp. right) real one-dimensional semitorus localized in the upper (resp. lower) half plane.

So, $M_{F_v^{+,T}}$ (resp. $M_{F_{\overline{v}}^{+,T}}$) decomposes according to a tower of conjugacy class representatives of $T_2(F_v^{+,T}) \subset \mathrm{GL}_2(F_{\overline{v}}^{+,T} \times F_v^{+,T})$ (resp. $T_2^t(F_{\overline{v}}^{+,T})$), characterized by increasing ranks, which are increasing integers modulo N , and localized respectively in the upper (resp. lower) half space.

As every left (resp. right) 1-dimensional real semitorus has the analytical development

$$\begin{aligned} T_L^1[n,m_n] &\simeq r(1,n,m_n)\; e^{2\pi inx} \;, \quad x \in F_v^{+,T} \;, \\ (\text{resp.} \quad T_R^1[n,m_n] &\simeq r(1,n,m_n)\; e^{-2\pi inx} \;), \end{aligned}$$

where $r(1,n,m_n) = (\lambda_+(n_N^2,m_N^2) - \lambda_-(n_N^2,m_N^2))/2$ according to proposition 2.15 and section 2.20, $M_{F_v^{+,T}}$ (resp. $M_{F_{\overline{v}}^{+,T}}$) will be developed analytically according to

$$\mathrm{ELLIP}_L(1,n,m_n) \simeq \mathop{\oplus}_{n=1}^{s} \mathop{\oplus}_{m_n} r(1,n,m_n)\; e^{2\pi inx} \;, \quad s \le \infty \;,$$

$$(\text{resp.} \quad \mathrm{ELLIP}_R(1,n,m_n) \simeq \mathop{\oplus}_{n=1}^{s} \mathop{\oplus}_{m_n} r(1,n,m_n)\; e^{-2\pi inx} \;, \quad s \le \infty \;)$$

as it will be seen in section 2.18.

Consequently, the analytical development of the representation space of $T_2(F_v^{+,T})$ (resp. $T_2^t(F_{\overline{v}}^{+,T})$) $\subset \mathrm{GL}_2(F_{\overline{v}}^{+,T} \times F_v^{+,T})$ is given by the Fourier series $\mathrm{ELLIP}_L(1,n,m_n)$ (resp. $\mathrm{ELLIP}_R(1,n,m_n)$) in such a way that each term of $\mathrm{ELLIP}_L(1,n,m_n)$ (resp. $\mathrm{ELLIP}_R(1,n,m_n)$) be the analytical representation of a conjugacy class representative of $T_2(F_v^{+,T})$ (resp. $T_2^t(F_{\overline{v}}^{+,T})$). By this way, the Fourier series receive an algebraic and geometric interpretation.

2.10 Irreducible representations entering into the Langlands two-dimensional bilinear correspondences

a) According to section 2.8, a representation space $\mathrm{Rep\ sp}(\mathrm{GL}_2(F_{\overline{\omega}}^T \times F_\omega^T))$ of the general bilinear semigroup $GL_2(F_{\overline{\omega}}^T \times F_\omega^T)$ is given by the bisemimodule $(M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}} \otimes M^{\mathrm{res}}_{F^T_{\omega_\oplus}})$ whose

analytic representation is given by $\mathrm{EIS}_R(2,n)\otimes\mathrm{EIS}_L(2,n)$. As the product, right by left, of the global Weil semigroups $W_{F_{\overline{\omega}}}\times W_{F_\omega}$ corresponds to the product, right by left, of the subgroups of Galois modular automorphisms of $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$ following section 1.7, we see that the sum of the products, right by left, of the equivalence classes of the irreducible 2-dimensional Weil-Deligne representation $\mathrm{Irr}\,\mathrm{Rep}^{(2)}(W_{F_{\overline{\omega}}}\times W_{F_\omega})$ of the bilinear global Weil semigroup $(W_{F_{\overline{\omega}}}\times W_{F_\omega})$ is given by $\mathrm{Rep\ sp}(\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega))$. On the other hand, $\mathrm{EIS}_R(2,n)\otimes\mathrm{EIS}_L(2,n)$ constitutes the sum of the products, right by left, of the equivalence classes of the irreducible cuspidal representation $\mathrm{Irr\,cusp}(\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega))$ of $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$.

b) Similarly, the representation space $\mathrm{Rep\ sp}(\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v))$ of $GL_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v)$ is the bisemimodule $(M_{F^{+,T}_{\overline{v}}}\otimes M_{F^{+,T}_v})$ whose analytic representation is given by $\mathrm{ELLIP}_R(1,n,m_n)\otimes\mathrm{ELLIP}_L(1,n,m_n)$. Let $W_{F_{\overline{v}}}\times W_{F_v}$ be the product, right by left, of the global Weil semigroups interpreted as the product, right by left, of the subgroups of Galois modular automorphisms of $\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v)$. Then, the sum of the products, right by left, of the equivalence classes of the representation $\mathrm{Irr}\,\mathrm{Rep}^{(1)}(W_{F_{\overline{v}}}\times W_{F_v})$ of the bilinear global Weil group $(W_{F_{\overline{v}}}\times W_{F_v})$ is given by $\mathrm{Rep\ sp}(\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v))$. On the other hand, $\mathrm{ELLIP}_R(1,n,m_n)\otimes\mathrm{ELLIP}_L(1,n,m_n)$ constitutes the sum of the products, right by left, of the equivalence classes of the irreducible cuspidal representation $\mathrm{Irr\,ELLIP}(\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v))$ of $\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v)$.

So, we can state the proposition :

2.11 Proposition

a) Over the sum of the products $F_{\overline{\omega}}\times F_\omega$ of the completions of the complex number semifields $\dot{\tilde{F}}_R$ and $\dot{\tilde{F}}_L$, there is a Langlands bilinear global correspondence given by the bijection:

$$\begin{aligned}\sigma_{F_{\overline{\omega}}\times F_\omega}:\quad & \mathrm{Irr}\,\mathrm{Rep}^{(2)}(W_{F_{\overline{\omega}}}\times W_{F_\omega})\longrightarrow\mathrm{Irr\,cusp}(\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega))\ ;\\ & \mathrm{Rep\ sp}(\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega))\longrightarrow\mathrm{EIS}_R(2,n)\times\mathrm{EIS}_L(2,n)\ .\end{aligned}$$

b) Over the sum of the products $F_{\overline{v}}\times F_v$ of the completions of the real number semifields $\dot{\tilde{F}}^+_R$ and $\dot{\tilde{F}}^+_L$, there is a Langlands bilinear global correspondence given by the bijection:

$$\begin{aligned}\sigma_{F_{\overline{v}}\times F_v}:\quad & \mathrm{Irr}\,\mathrm{Rep}^{(1)}(W_{F_{\overline{v}}}\times W_{F_v})\longrightarrow\mathrm{Irr\,ELLIP}(\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v))\ ;\\ & \mathrm{Rep\ sp}(\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v))\longrightarrow\mathrm{ELLIP}_R(1,n,m_n)\times\mathrm{ELLIP}_L(1,n,m_n)\ .\end{aligned}$$

Proof. Taking into account the section 2.10, the Langlands global correspondence $\sigma_{F_{\overline{\omega}}\times F_\omega}$ implies that, to the double symmetric tower of conjugacy class representatives of $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$ corresponds the double symmetric tower of the cuspidal representations of these conjugacy class representatives consisting in terms of the Fourier developments of the considered cusp forms.
More concretely, let $g^{(2)}_{R\times L}[n]$ be a conjugacy class representative of $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$, i.e. a $B_{F_{\overline{\omega}_n}}\otimes B_{F_{\omega_n}}$-subbisemimodule $M_{F_{\overline{\omega}_n}}\otimes M_{F_{\omega_n}}\subset M_{F_{\overline{\omega}}}\otimes M_{F_\omega}$ and let $\mathrm{eis}_{R\times L}(2,n)=\lambda(2,n)\cdot e^{-2\pi inz}\otimes\lambda(2,n)\cdot e^{2\pi inz}\in \mathrm{EIS}_R(2,n)\otimes\mathrm{EIS}_L(2,n)$ denote its analytic cuspidal (toroidal) counterpart.
Then, we have the following set of bijections:

$$\begin{array}{ccc}
g^{(2)}_{R\times L}[1] & \xrightarrow{\sim} & \mathrm{eis}_{R\times L}(2,1)\ , \\
\vdots & & \vdots \\
g^{(2)}_{R\times L}[n] & \xrightarrow{\sim} & \mathrm{eis}_{R\times L}(2,n)\ , \\
\vdots & & \vdots \\
g^{(2)}_{R\times L}[s] & \xrightarrow{\sim} & \mathrm{eis}_{R\times L}(2,s)\ ,
\end{array}$$

implying the bijection

$$\mathrm{Rep\ sp}(\mathrm{GL}_2(F_{\overline{\omega}_\oplus}\times F_{\omega_\oplus}))\ \xrightarrow{\sim}\ \mathrm{EIS}_{R\times L}(2,n)=\mathrm{EIS}_R(2,n)\times\mathrm{EIS}_L(2,n)\ ,$$

where:

- $\mathrm{Rep\ sp}(\mathrm{GL}_2(F_{\overline{\omega}_\oplus}\times F_{\omega_\oplus}))=\bigoplus\limits_{n=1}^{s}\mathrm{Rep\ sp}(g^{(2)}_{R\times L}[n])$;
- $\mathrm{EIS}_{R\times L}(2,n)=\bigoplus\limits_{n=1}^{s}\mathrm{eis}_{R\times L}(2,n)$.

The second Langlands global correspondence $\sigma_{F_{\overline{v}}\times F_v}$ can be handled similarly. ■

Note that this way of introducing two-dimensional Langlands correspondences by global class field concepts corresponds to a new approach of this problem which was developed locally more particularly by G. Henniart [Hen], H. Carayol [Car], B. Conrad, F. Diamond and R. Taylor [C-D-T], S. Gelbart [Ge] and M. Harris and R. Taylor [H-T].

2.12 Representations of products of Hecke operators

The ring of endomorphisms of the $GL_2(F_{\overline{\omega}}\times F_\omega)$-bisemimodule $(M_{F_{\overline{\omega}}}\otimes M_{F_\omega})$, decomposing it into the set of subbisemimodules $(M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F_{\omega_{n,m_{\omega_n}}}})$ following the bisublattices

$(\Lambda^{(2)}_{\overline{\omega}_{n,m_{\omega_n}}} \otimes \Lambda^{(2)}_{\omega_{n,m_{\omega_n}}})$, is generated over $\mathbb{Z}/N\,\mathbb{Z}$ by the products $(T_{q_R} \otimes T_{q_L})$ of Hecke operators T_{q_R} and T_{q_L} for $q \nmid N$ and by the products $(U_{q_R} \otimes U_{q_L})$ of Hecke operators U_{q_R} and U_{q_L} for $q \mid N$ [M-W]: it is noted $T_H(N)_R \otimes T_H(N)_L$.

Note that the ring of endomorphisms of the real $\mathrm{GL}_2(F^+_{\overline{v}} \times F^+_v)$-bisemimodule $(M_{F^+_{\overline{v}}} \times M_{F^+_v})$ can also be defined classically by products, right by left, of Hecke operators, if we take into account the proposition 2.5 and corollary 2.6.

The coset representative of U_{q_L} , referring to the upper half plane, can be chosen to be upper triangular and given by the integral matrix $\left(\begin{smallmatrix} 1 & b_N \\ 0 & q_N \end{smallmatrix}\right)$ of the congruence subgroup $\Gamma_L(N)$ in $\mathrm{GL}_2(\mathbb{Z})$ such that:

- $q_N = * + q \cdot N$;

  $b_N = * + b \cdot N$,

  where $*$ denotes an integer inferior to N .

  Indeed, $q_N = q \cdot N$ involves that $q \mid q_N$ and also that $q \mid N$ if $N = rq$, $r \in \mathbb{N}$.

  On the other hand, $q \nmid N$ is equivalent to the condition $q \nmid q_N$ since then, $q_N = q \cdot N + s$.

- q is the cardinality of the infinite place v_q ;

- b refers to the multiplicity of the Hecke sublattice of level q .

Similarly, the coset representative of U_{q_R} , referring to the lower half plane, can be chosen to be lower triangular and given by the integral matrix $\left(\begin{smallmatrix} 1 & 0 \\ b_N & q_N \end{smallmatrix}\right)$ of the congruence subgroup $\Gamma_R(N)$ in $\mathrm{GL}_2(\mathbb{Z})$.

For general n , we would have respectively integral matrices $\left(\begin{smallmatrix} a_N & b_N \\ 0 & d_N \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_N & 0 \\ b_N & d_N \end{smallmatrix}\right)$ of determinant $a_N \cdot d_N = * + n \cdot N$.

In fact, the congruence subgroup $\Gamma_L(N)$ denotes a general inverse image [Hus] of the congruence subgroups $\Gamma(N)$, $\Gamma_1(N)$ or $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z}/N\,\mathbb{Z})$ [M-W], [La], denoting the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with the suitable congruences and acting as group(s) of symmetries of the hyperbolic upper half plane by the rule $z \to (a\ z+b)/(c\ z+d)$, $z \in \mathbb{C}$ being a point of order N , $\mathrm{Im}(z) > 0$.

The congruence subgroup $\Gamma_R(N)=\Gamma_L(N)^t$ denotes the general inverse image of the group of matrices $\left(\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right)$ with the suitable congruences acting as group of symmetries of the hyperbolic lower half plane by the rule $z^*\to(a\ z^*+b)\big/(c\ z^*+d)$, $z^*\in\mathbb{C}$, $\mathrm{Im}(z^*)<0$.

Considering that the group of matrices

$$u_2(b_N)=\begin{pmatrix}1 & b_N\\ 0 & 1\end{pmatrix}\qquad\text{and}\qquad u_2(b_N)^t=\begin{pmatrix}1 & 0\\ b_N & 1\end{pmatrix}\ ,$$

elements of the nilpotent group relative to the v_q-th infinite place, generates $\mathbb{F}_q$, the following coset representative

$$g_2(q_N^2,b_N)=\left[\begin{pmatrix}1 & b_N\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_N & 1\end{pmatrix}\right]\begin{pmatrix}1 & 0\\ 0 & q_N^2\end{pmatrix}$$

will be adopted for $U_{q_R}\otimes U_{q_L}$, where $\left(\begin{smallmatrix}1 & 0\\ 0 & q_N^2\end{smallmatrix}\right)$ is the element of the split Cartan subgroup referring to a pseudo-ramified quadratic infinite place $v_{q^2}\equiv\overline{v}_q\times v_q$.

$g_2(q_N^2,b_N)$ has a Gauss decomposition form in diagonal and unipotent parts. The unipotent part of $g_2(q_N^2,b_N)$ is $u_2(b_N)\cdot u_2(b_N)^t$ (and not $u_2(b_N)^t\cdot u_2(b_N)$), since we are dealing with (bi)linear functionals according to the Riesz lemma, and corresponds to the element of the decomposition group associated with the split Cartan subgroup element $\alpha_{q^2}=\left(\begin{smallmatrix}1 & 0\\ 0 & q_N^2\end{smallmatrix}\right)$.

2.13 Hecke eigenvalues and decomposition group eigenvalues coincide

The **decomposition group** $D_{q_N^2}$ of the square of q_N associated with the split Cartan subgroup element $\alpha_{q_N^2}=\left(\begin{smallmatrix}1 & 0\\ 0 & q_N^2\end{smallmatrix}\right)$ is given by $\left\{D_{q_N^2;b_N}=u_2(b_N)\cdot u_2(b_N)^t\right\}_{b_N}$. Indeed, the semisimple form of $D_{q_N^2;b_N}$ is unimodular since $\det(\lambda^+_{D_{q_N^2;b_N}}\cdot\lambda^-_{D_{q_N^2;b_N}})=1$ where $\lambda^\pm_{D_{q_N^2;b_N}}$ are the two eigenvalues of $D_{q_N^2;b_N}$ which thus maps q_N^2 into itself.

Let $D_{q_N^2;b_N}$ be the element of the decomposition group acting on the split Cartan subgroup element $\alpha_{q_N^2}$. Let $\lambda_+(q_N^2,b_N^2)$ and $\lambda_-(q_N^2,b_N^2)$ be the eigenvalues of $(D_{q_N^2;b_N^2}\cdot\alpha_{q_N^2})$ so that the determinant of the semisimple form of $(D_{q_N^2,b_N^2}\cdot\alpha_{q_N^2})$ be given by $\det(D_{q_N^2;b_N^2}\cdot$

$\alpha_{q_N^2})_{ss} = \lambda_+(q_N^2, b_N^2) \cdot \lambda_-(q_N^2, b_N^2)$. Then, the decomposition group $D_{q_N^2;b_N^2}$ is such that:

$$\begin{aligned} D_{q_N^2;b_N^2} : \quad \det(\alpha_{q_N^2}) &\longrightarrow \det(D_{q_N^2;b_N^2} \cdot \alpha_{q_N^2})_{ss} \\ q_N^2 &\longrightarrow q_N^2 \end{aligned}$$

where $(\ )_{ss}$ denotes the corresponding semisimple form.

2.14 Definition: The Frobenius element Frob(q^2)

In the pseudo-unramified case, i.e. when $N = 1$, $D_{q^2,b^2} = \mathrm{Frob}(q^2)$ which gives:

$$\begin{aligned} D_{q^2,b^2} : \quad \det(\alpha_{q^2}) &\longrightarrow \det(\alpha_{q^2} \cdot \mathrm{Frob}(q^2)) \\ q^2 &\longrightarrow q^2 \end{aligned}$$

mapping the square of the global residue field class degree $f_{v_q} = q$ into itself.

So, **the Hecke eigenvalues and the Frobenius eigenvalues also coincide** in the pseudo-unramified case.

As a result of the coincidence of the Hecke eigenvalues with the decomposition group eigenvalues or with the Frobenius eigenvalues, we can then state the following proposition and corollary:

2.15 Proposition

There is an explicit irreducible semisimple (pseudo-)ramified representation, associated with a weight two cusp form

$$\rho_{\lambda_\pm} : \quad \mathrm{Gal}(\widetilde{F}_{\overline{v}}^+/k) \times \mathrm{Gal}(\widetilde{F}_v^+/k) \longrightarrow \mathrm{GL}_2(T_H(N)_R \otimes T_H(N)_L) ,$$

having eigenvalues:

$$\lambda_\pm(q_N^2, b_N^2) = \frac{(1 + b_N^2 + q_N^2) \pm [(1 + b_N^2 + q_N^2)^2 - 4q_N^2]^{\frac{1}{2}}}{2}$$

verifying

$$\begin{aligned} \mathrm{trace}\, \rho_{\lambda_\pm} &= 1 + b_N^2 + q_N^2 , \\ \det \rho_{\lambda_\pm} &= \lambda_+(q_N^2, b_N^2) \cdot \lambda_-(q_N^2, b_N^2) = q_N^2 . \end{aligned}$$

Proof. Indeed, we have a representation $\rho_{\lambda_\pm}$ whose coset representatives are given by the matrices:

$$g_2(q_N^2,b_N)=\left[\begin{pmatrix}1 & b_N\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_N & 1\end{pmatrix}\right]\begin{pmatrix}1 & 0\\ 0 & q_N^2\end{pmatrix}=\begin{pmatrix}1+b_N^2 & b_Nq_N^2\\ b_N & q_N^2\end{pmatrix}$$

having eigenvalues $\lambda_\pm(q_N^2,b_N^2)$.
It is then easy to check that trace $\rho_{\lambda_\pm}=1+b_N^2+q_N^2$ and that $\det\rho_{\lambda_\pm}=q_N^2$. ■

2.16 Corollary

Let $\rho^{nr}_{\lambda_\pm}$ be the corresponding irreducible semisimple (pseudo-)unramified representation, associated with a weight two cusp form:

$$\rho^{nr}_{\lambda_\pm}:\quad \mathrm{Gal}(\widetilde F^{+,nr}_{\overline v}/k)\times\mathrm{Gal}(\widetilde F^{+,nr}_{v}/k)\longrightarrow \mathrm{GL}_2(T_{H_R}\otimes T_{H_L})\ ,$$

where

$$T_{H_R}\otimes T_{H_L}=T_H(N)_R\otimes T_H(N)_L|_{N=1}\ ,$$

having eigenvalues:

$$\lambda^{nr}_\pm(q^2,b^2)=\frac{(1+b^2+q^2)\pm[(1+b^2+q^2)^2-4q^2]^{\frac12}}{2}$$

verifying

$$\begin{aligned}\mathrm{trace}\,\rho^{nr}_{\lambda_\pm}(\mathrm{Frob}\,q^2)&=1+b^2+q^2\ ,\\ \det\rho^{nr}_{\lambda_\pm}(\mathrm{Frob}\,q^2)&=\lambda^{nr}_+(q^2,b^2)\cdot\lambda^{nr}_-(q^2,b^2)=q^2\ .\end{aligned}$$

Then, $\rho^{nr}_{\lambda_\pm}$ has a characteristic polynomial having the form

$$X^2-\mathrm{trace}\,\rho^{nr}_{\lambda_\pm}(\mathrm{Frob}\,q^2)X+q^2=0$$

where X is an indeterminate [La-Tr], [Shi], [Ser1], [Clo].

2.17 On the relevance of considering bialgebras for modular representations

Assume classically that f is a normalized eigenform associated with the congruence subgroup $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$ of weight $k\geq 2$. Then, T_n is the Hecke operator verifying

$T_n f = c(n,f)f$, for each integer n , where $c(n,f)$ is an algebraic integer. Let K_f be the number field generated over $\mathbb{Q}$ by the $\{c(n,f)\}$ and θ its ring of integers.

The normalized eigenforms f , expanded in formal power series $f = \sum\limits_n a_n q^n$, are non-zero cusp forms of the space $S(N)$ and are eigenvectors for all the T_n , satisfying $a_1 = 1$ and $a_n = c(n,f)$: so, Fourier coefficients of f and eigenvalues of T_n coincide. The space $S(N)$ of the $f = \sum\limits_n c(n,f)q^n$ is thus a [semi-]algebra [Pie3] of cusp forms.

If H denotes the Poincare upper half plane in $\mathbb{C}$, the eigenforms f , elements of the [semi-]algebra $S(N)$, are holomorphic in H and defined in $\{\mathrm{Im}(z) > 0\}$ with respect to the variable $z \in \mathbb{C}$ of $q = e^{2\pi iz}$. The dual [semi-]algebra of the [semi-]algebra $S(N)$, relabelled $S_L(N)$, is defined as the right [semi-]algebra $S_R(N)$ whose elements are normalized eigenforms f_R associated with the congruence subgroup $\Gamma_1(N)^t$ (of the transposed matrices of $\Gamma_1(N)$) and defined in the Poincare lower half plane H^* . ("L " is for left and refers to the upper half plane while " R " is for right and refers to the lower half plane).

These eigenforms f_R of $S_R(N)$, expanded in Fourier series $f_R = \sum\limits_n a_{n,R} q_R^n = \sum\limits_n a_{n,R} e^{-2\pi inz}$, are holomorphic in H^* and defined in $\{\mathrm{Im}(z) < 0\}$. The f_R are eigenfunctions of Hecke operators $T_{n,R}$, defined with respect to the congruence subgroup $\Gamma_1(N)^t$ and verifying

$$T_{n,R} f_R = c(n, f_R) f_R \qquad \text{with} \quad a_{n,R} = c(n, f_R) \ .$$

The bi[semi]algebra $S_L^e(N)$ associated with the [semi-]algebra $S_L(N)$ is given by $S_L^e(N) = S_R(N) \otimes_\theta S_L(N)$ and is of special importance owing to the natural homomorphism $\psi : S_L^e(N) \to \mathrm{End}_\theta(S_L(N))$ where $\mathrm{End}_\theta(S_L(N))$ is isomorphic to the Hecke algebra $\mathcal{H}_L$ generated by the Hecke operators $T_{n,L}(N)$ [F-D].

Considering the isomorphism

$$\omega : \quad \mathrm{End}_\theta(S_R(N)) \otimes \mathrm{End}_\theta(S_L(N)) \longrightarrow \mathrm{End}_\theta(S_R(N) \otimes S_L(N))$$

where $\mathrm{End}_\theta(S_R(N)) \otimes \mathrm{End}_\theta(S_L(N))$ is isomorphic to the product $\mathcal{H}_R \otimes \mathcal{H}_L$ of the Hecke algebras $\mathcal{H}_R$ and $\mathcal{H}_L$, generated by the Hecke operators $T_{n,R}$ and $T_{n,L}$, while $\mathrm{End}_\theta(S_R(N) \otimes S_L(N))$ is isomorphic to the Hecke bialgebra $\mathcal{H}_{R\otimes L}$, generated by tensor products $T_{n,R} \otimes T_{n,L}$ of Hecke operators $T_{n,R}$ and $T_{n,L}$ acting on $S_L^e(N)$, we are led to the following

commutative diagram:

$$\begin{array}{ccc} S^e_L(N) & \longrightarrow & \mathrm{End}_\theta(S_L(N)) \\ & \searrow & \uparrow \\ & & \mathrm{End}_\theta(S^e_L(N)) \end{array}$$

It then becomes clear that $\mathrm{End}_\theta(S_L(N))$ will be worked out successfully by taking into account the endomorphisms $\mathrm{End}_\theta(S^e_L(N))$ of the bialgebra $S^e_L(N)$, i.e. by considering the Hecke bialgebra $\mathcal{H}_{R\otimes L}$ of tensor products $T_{n,R}\otimes T_{n,L}$ of right Hecke operators $T_{n,R}$ by left Hecke operators $T_{n,L}$ acting on tensor products of cusp forms $f_R\otimes f_L = \sum\limits_n a_{n,R}q^n_R \otimes \sum\limits_n a_{n,L}q^n_L \in S_R(N)\otimes S_L(N)$.

$f_R\otimes_D f_L$ is called a modular biform and is naturally defined in the present context on the complex algebraic bilinear semigroup $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$.
$f_R\otimes_D f_L\in C^\infty(\mathrm{GL}_2(F^T_R\times F^T_L)\big/\mathrm{GL}_2((\mathbb{Z}/N\cdot\mathbb{Z})^2$.

2.18 The space of global elliptic semimodules

According to section 2.7, the compactified bisemispace

$$\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}} = P_2(F^T_{[\overline{\omega}_1]}\times F^T_{[\omega_1]})\setminus \mathrm{GL}_2(F^T_R\times F^T_L)\Big/\mathrm{GL}_2(\mathbb{Z}/N\cdot\mathbb{Z})^2$$

decomposes into pairs of two-dimensional semitori following the "modular" conjugacy classes of $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$ which are in one-to-one correspondence with the places of $F^T_{\overline{\omega}}$ or F^T_ω counted with their multiplicities.

Remark that there is a one-to-one correspondence between $\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}|L}}$, restricted to the upper half space (left case), and given by the semispace

$$\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}|L}} = P_2(F^T_{[\omega_1]})\setminus T_2(F^T_L)\Big/T_2(\mathbb{Z}/N\cdot\mathbb{Z}) ,$$

and the jacobian

$$J(N)_{\mathbb{C}} = H^0(X(N)_{\mathbb{C}},\Omega)\Big/H_1(X(N)_{\mathbb{C}},\mathbb{Z})$$

of the Riemann surface $X(N)_{\mathbb{C}}$ corresponding to the group $\Gamma(N)$ associated with$\Gamma_L(N)$ (see section 2.12).

Indeed, the subgroup $T_2(\mathbb{Z}/N\,\mathbb{Z})$ is a representation of Hecke lattice operators cutting $T_2(F^T_L)$ following its modular conjugacy classes which are in one-to-one correspondence with the sublattices or periods of the jacobian $J(N)_{\mathbb{C}}$ [Hin], [Bos]. In order to have a natural model over $\mathbb{Q}$ of $X(N)_{\mathbb{C}}$ leading to $J(N)_{\mathbb{Q}} = \mathrm{Pic}(X(N)_{\mathbb{Q}})$, the boundary

$$\partial\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}} = P_2(F^{+,T}_{[\overline{v}_1]}\times F^{,T}_{[v_1]})\setminus \mathrm{GL}_2(F^{+,T}_R\times F^{+,T}_L)\Big/\mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2$$

of $\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$ is assumed to be decomposed into pairs of packets of one-dimensional semitori following the biplaces of $F^{+,T}_{[\overline{v}]}\times F^{+,T}_{[v]}$ such that each packet of $1D$-semitori covers the corresponding complex representative of the considered conjugacy class of $T_2(F^T_\omega)$ or of $T^t_2(F^T_{\overline{\omega}})$.

Let the number of infinite complex places be equal to the number of real infinite places. Then, every $2D$-semitorus $T^2_L[n,m_n=1]\in\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_{N|L}}}}$ (resp. $T^2_R[n,m_n=1]\in\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_{N|R}}}}$) can be decomposed into a packet of $1D$-semitori $T^1_L[n,m_n]$ (resp. $T^1_R[n,m_n]$) referring to the n-th place of $F^{+,T}_{[v]}$ (resp. $F^{+,T}_{[\overline{v}]}$) in one-to-one correspondence with the n-th modular conjugacy class of $\mathrm{GL}_2(F^{+,T}_{[\overline{v}]}\times F^{+,T}_{[v]})$ corresponding to the n-th coset of the boundary $\partial\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$ of $\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$.

So, every $2D$-semitorus of $\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$ can be decomposed into a packet $\{T^1_L[n,m_n]\}_{m_n}$ of $1D$-semitori isomorphic to a set of affine curves of the $T_2(F^+_{v_n})$-subsemimodule $M_{F^+_{v_n}}$ and characterized each one by an integer " n " with respect to the representation of the smallest pseudo-ramified normal subsemigroup of class " 1 " of $T_2(F^+_v)$ which is a $P_2(F^+_{[v_1]})$-subsemimodule $M^I_{[v_1]}$ also called a left quantum. The integer n corresponds to the global class residue degree

$$[F^{+,nr}_{v_n}:k]=f_{v_n}=n$$

in such a way that a pseudo-ramified one-dimensional semitorus of class n has a rank given by $[F^+_{v_n}:k]=n\cdot N$ where N is the rank of the quantum $M^I_{[v_1]}$.

The set of continuous complex-valued functions over the conjugacy class representatives of the $T_2(F^{+,T}_v)$-semimodule $M_{F^{+,T}_v}$ (resp. $T^t_2(F^{+,T}_{\overline{v}})$-semimodule $M_{F^{+,T}_{\overline{v}}}$) is a semisheaf of rings whose set of sections is noted $A_L\equiv\Gamma(M_{F^{+,T}_v})$ (resp. $A_R\equiv\Gamma(M_{F^{+,T}_{\overline{v}}})$).

This set of sections A_L (resp. A_R) has the structure of a semiring $C(M^I_{[v_1]})$ (resp. $C(M^I_{[\overline{v}_1]})$) of continuous complex valued functions over left (resp. right) quanta $M^I_{[v^1_n]}$ (resp. $M^I_{[\overline{v}^1_n]}$) due to the injective maps:

$$\begin{aligned}
m^I_L(n):\quad & M^I_{[v_1]}\longrightarrow M_{F^{+,T}_{v_n}}\ ,\quad 1\le n\le s\ ,\\
\text{(resp.}\quad m^I_R(n):\quad & M^I_{[\overline{v}_1]}\longrightarrow M_{F^{+,T}_{\overline{v}_n}}\ ,\quad 1\le n\le s\ ),
\end{aligned}$$

where $M_{F^{+,T}_{v_n}}$ (resp. $M_{F^{+,T}_{\overline{v}_n}}$) is a left- (resp. right)-$T_2(F^{+,T}_{v_n})$-subsemimodule $\in M_{F^{+,T}_v}$ (resp. $T^t_2(F^{+,T}_{\overline{v}_n})$-subsemimodule $\in M_{F^{+,T}_{\overline{v}}}$).

For every section $(s_L)_{n,b}\in A_L$ (resp. $(s_R)_{n,b}\in A_R$), let $\mathrm{End}_{F^{+,T}_v}(A_L)$ (resp.

$\mathrm{End}_{F^{+,T}_{\overline{v}}}(A_R)$) be the Frobenius endomorphism of A_L (resp. A_R) and let

$$q\Big/\mathbb{Q}_L \longrightarrow q^n/\mathbb{Q}_L \in \mathrm{End}_{F^{+,T}_{v_n}}(s_L)_{n,b}$$
$$(\text{resp.}\quad q\Big/\mathbb{Q}_R \longrightarrow q^n/\mathbb{Q}_R \in \mathrm{End}_{F^{+,T}_{\overline{v}_n}}(s_R)_{n,b}\ )$$

be the Frobenius substitution where $q^n/\mathbb{Q}_L = e^{2\pi inx}$ (resp. $q^n/\mathbb{Q}_R = e^{-2\pi inx}$); $x \in F^{+,T}_v$ being a point of order N .

A global elliptic left (resp. right) -A_L-semimodule $\phi_L(s_L)$ (resp. A_R-semimodule $\phi_R(s_R)$) over $\mathbb{Q}$ is a ring homomorphism:

$$\phi_L : \quad A_L \longrightarrow \mathrm{End}_{F^{+,T}_v}(A_L)$$
$$(\text{resp.}\quad \phi_R : \quad A_R \longrightarrow \mathrm{End}_{F^{+,T}_{\overline{v}}}(A_R)\ )$$

given, for the sections $s_L \in A_L$ (resp. s_R in A_R), by:

$$\phi_L(s_L) = \sum_{n=1}^{s} \sum_b \phi(s_L)_{n,b}\, q^n\Big/\mathbb{Q}_L\ , \qquad s \leq \infty\ ,$$
$$(\text{resp.}\quad \phi_R(s_R) = \sum_{n=1}^{s} \sum_b \phi(s_R)_{n,b}\, q^n\Big/\mathbb{Q}_R\ , \qquad s \leq \infty\ ),$$

where

- $\sum\limits_n$ runs over the classes " n " of the set of pairs of $1D$-semitori isomorphic to the set of pairs of the affine curves of $\mathrm{GL}_2(F^+_R \times F^+_L)\big/\,\mathrm{GL}_2(\mathbb{Z}\big/N\ \mathbb{Z})^2$;

- $\sum\limits_b$ runs over the representatives of the pairs of the $1D$-semitori of class " n ". These representatives correspond to the number of (semi)orbits or ideals of the decomposition group $D_{n^2_N;b_N}$ according to definition 2.13;

- $\phi(s_L)_{n,b}$ (resp. $\phi(s_R)_{n,b}$) is a coefficient of the Fourier series $\phi_L(s_L)$ (resp. $\phi_R(s_R)$).

Remark that the name of global elliptic left (resp. right) A_L-semimodule $\phi_L(s_L)$ (resp. A_R-semimodule $\phi_R(s_R)$), also noted $\mathrm{ELLIP}_L(1,n,m_n)$ (resp. $\mathrm{ELLIP}_R(1,n,m_n)$) in section 2.11 for these (truncated) Fourier series is justified by the facts that:

a) a subset of $1D$-semitori of level p , where p is a prime number, can be identified with the orbit space of an elliptic curve $E(\mathbb{F}_p)$ under the action of the decomposition group $D_{p^2_N}$ as it will be seen in chapter 3;

b) these global elliptic (semi)modules have some analogy of construction with respect to the local elliptic modules (over function fields) introduced by V. Drinfeld [Drin] (see also [And]).

2.19 Proposition

Let $S_L(f)$ (resp. $S_R(f)$) be the left [semi-]algebra (resp. right [semi-]algebra) of modular forms $f_L(z)=\sum\limits_n a_{n,L}\,q_L^n$ (resp. $f_R(z)=\sum\limits_n a_{n,R}\,q_R^n$) $q_L=e^{2\pi iz}$ (resp. $q_R=e^{-2\pi iz}$), $z\in\mathbb{C}$,

- *being normalized eigenforms of Hecke operators associated with the congruence subgroup $\Gamma_L(N)$ (resp. $\Gamma_R(N)$) introduced in section 2.12;*

- *characterized by a weight two and a level N .*

On the other hand, let $S_L(\phi)$ (resp. $S_R(\phi)$) be the left [semi-]algebra (resp. right [semi-]algebra) of global elliptic A_L-semimodules $\phi_L(s_L)$ (resp. $\phi_R(s_R)$) in the sense that $f_L(z)\simeq\phi_L(s_L)$ (resp. $f_R(z)\simeq\phi_R(s_R)$). Then, we have the following inclusions of left [semi-]algebras (resp. right [semi-]algebras):

$$\begin{aligned} & S_L(\phi)\hookrightarrow S_L(f) \\ \text{(resp.}\quad & S_R(\phi)\hookrightarrow S_R(f)\ ). \end{aligned}$$

Proof: according to section 2.8, the (truncated) Eisenstein series $\mathrm{EIS}_L(2,n,m_n=1)$ (resp. $\mathrm{EIS}_R(2,r,m_n=1)$), which is a (pseudo)modular form $f_L(z)$ (resp. $f_R(z)$) [Ma3], constitutes the analytic development of the $T_2(F_\omega^T)$-semimodule $M^{\mathrm{res}}_{F_\omega^T}$ (resp. $T_2^t(F_{\overline{\omega}}^T)$-semimodule $M^{\mathrm{res}}_{F_{\overline{\omega}}^T}$) according to the global Langlands correspondence a) of proposition 2.11. On the other hand, the global elliptic A_L-semimodule $\phi_L(s_L)$ (resp. A_R-semimodule $\phi_R(s_R)$) constitutes the analytic development of the $T_2(F_v^{+,T})$-semimodule $M_{F_v^{+,T}}$ (resp. $T_2(F_{\overline{v}}^{+,T})$-semimodule $M_{F_{\overline{v}}^{+,T}}$) according to section 2.10 and the global correspondence of Langlands b) of proposition 2.11.
Then, if we assume that (conditions of proposition 2.5):

1. the complex places of F_ω^T have simple multiplicities given by $m_n=1$;

2. the number of complex places of F_ω^T is equal to the number of real places of $F_v^{+,T}$;

3. $\partial\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}\hookrightarrow\overline{S}^{P_2}_{\mathrm{GL}_{2_{\mathbb{Z}_N}}}$ (see section 2.18), implying that complex irreducible completions coincide with real irreducible completions,

it is then clear that

- $f_L(z) \simeq \phi_L(s_L)$ (resp. $f_R(z) \simeq \phi_R(s_R)$);

- $S_L(\phi) \hookrightarrow S_L(f)$ (resp. $S_R(\phi) \hookrightarrow S_R(f)$). ■

2.20 Supercuspidal representations in terms of global elliptic bisemimodules

Let $\otimes_D$ denote a diagonal tensor product, i.e. a tensor product whose only diagonal terms with respect to a basis $\{e_{n,b}\otimes e_{n,b}\}$ are different from zero. Let $\lambda_+(q_N^2,b_N^2)$ and $\lambda_-(q_N^2,b_N^2)$ be the eigenvalues of the products of Hecke operators having as coset representatives $g_2(q_N^2,b_N)$.

Assume the existence of a global elliptic A_{R-L}-bisemimodule $\phi_R(s_R)\otimes_D\phi_L(s_L) = \sum\limits_{n,b}\phi(s_R)_{n,b}q^n_{/\mathbb{Q}\,R}\otimes_D\sum\limits_{n,b}\phi(s_L)q^n_{/\mathbb{Q}\,L}$ whose coefficients $\phi(s_L)_{n,b}$ (resp. $\phi(s_R)_{n,b}$)are given by $\phi(s_L)_{n,b} = (\lambda_+(n_N^2,b_N^2))$ (resp. $\phi(s_R)_{n,b} = (\lambda_-(n_N^2,b_N^2))$) and which verifies $f_R(z)\otimes_D f_L(z) \simeq \phi_R(s_R)\otimes_D\phi_L(s_L)$, i.e. the "inclusion", in the sense of 2.19, of the elliptic A_{R-L}-bisemimodule into a diagonal tensor product of weight two cusp forms. Consider the isomorphism:

$$\begin{aligned} i_R\otimes i_L:\quad \phi_R(s_R)\otimes_D\phi_L(s_L) &= \sum_{n,b}(\lambda_+(n_N^2,b_N^2))q^n_{/\mathbb{Q}\,R}\otimes_D\sum_{n,b}(\lambda_-(n_N^2,b_N^2))q^n_{/\mathbb{Q}\,L}\\ \longrightarrow \widetilde\phi_R(s_R)\otimes_D\widetilde\phi_L(s_L) &= \sum_{n,b}(r(n_N^2,b_N^2))q^n_{/\mathbb{Q}\,R}\otimes_D\sum_{n,b}(r(n_N^2,b_N^2))q^n_{/\mathbb{Q}\,L}\end{aligned}$$

sending the eigenvalues $(\lambda_\pm(n_N^2,b_N^2))$ to $(r(n_N^2,b_N^2))$ defined by $(r(n_N^2,b_N^2)) = ((\lambda_+(n_N^2,b_N^2))+(\lambda_-(n_N^2,b_N^2))/2 = \operatorname{trace}\rho_{\lambda_\pm}(n_N^2,b_N^2)/2$ given in proposition 2.15.

Then, $\widetilde\phi_R(s_R)\otimes_D\widetilde\phi_L(s_L)$ decomposes into a sum of (tensor) products of irreducible semitoric curves localized in the upper and in the lower half space corresponding to each other by pairs of same class n and same value of b in such a way that each pair of semitoric curves be characterized by a radius $(r(n_N^2,b_N^2))$ and a center at the origin. The isomorphism $i_R\otimes i_L$ has then to be interpreted as a translation of the toric curves since $(\lambda_\pm(n_N^2,b_N^2))$ may be viewed as a pair $\{\mathrm{cent}(n_N^2,b^2)),(r(n_N^2,b_N^2))\}$ where $(\mathrm{cent}(n_N^2,b_N^2)) = ((\lambda_+(n_N^2,b_N^2))-(\lambda_-(n_N^2,b_N^2))/2$ is the image of the translated center under $i_{R,L}$. It then results that the eigenvalues $\lambda_+(n_N^2,b_N^2)$ and $\lambda_-(n_N^2,b_N^2)$ are equivalent.

A diagonal tensor product of weight two cusp forms thus has for representation $\widetilde{\phi}_R(s_R)\otimes_D\widetilde{\phi}(s_L)$ which will be rewritten according to:

$$\widetilde{\phi}_R(s_R)\otimes_D\widetilde{\phi}(s_L)\Rrightarrow\Pi_R\otimes_D\Pi_L=\sum_n(m_R(n)\Pi_R(n))\otimes_D(m_L(n)\Pi_L(n))$$

where $m_L(n)$ (resp. $m_R(n)$) denotes the multiplicity of the irreducible curve $\Pi_L(n)$ (resp. $\Pi_R(n)$) of global level n , i.e. associated with the global left (resp. right) place v_n (resp. $\overline{v}_n$).

2.21 Proposition

Let $GL_2(F_{\overline{v}}^+\times F_v^+)$ be the general bilinear algebraic semigroup over $F_{\overline{v}}^+\times F_v^+$. Then, $\mathrm{GL}_2(F_{\overline{v}}^+\times F_v^+)$ *has for irreducible supercuspidal representation the Grothendieck group* $\mathrm{Groth}(\mathrm{GL}_2(F_{\overline{v}}^+\times F_v^+))$ *defined by*

$$\mathrm{Groth}(GL_2(F_{\overline{v}}^+\times F_v^+))=\Pi_R\otimes_D\Pi_L=\sum_n(m_R(n)\Pi_R(n))\otimes_D(m_L(n)\Pi_L(n))\ .$$

Proof. Indeed $\Pi_R\otimes_D\Pi_L$ constitutes an irreducible supercuspidal representation of $GL_2(F_{\overline{v}}^+\times F_v^+)$ such that the sum of the representations of the conjugacy classes of $\mathrm{GL}_2(F_{\overline{v}}^+\times F_v^+)$ are in bijection with the sum of the irreducible supercuspidal subrepresentations of $\Pi_R\otimes_D\Pi_L$ over the quadratic places $(\overline{v}_n\times v_n)$. ■

2.22 Injective nilpotent morphisms

Assume that the bi[semi]algebra $S_R(\phi)\otimes_D S_L(\phi)$ of elliptic A_{R-L}-bisemimodules is included into the bi[semi]algebra $S_R(f)\otimes_D S_L(f)$ of weight two cusp forms.

Let $\Pi^0_{R\times L}:GL_2(F_{\overline{v}}^+\times F_v^+)\to\phi^0_R(s_R)\otimes_D\phi^0_L(s_L)$ denote a simple supercuspidal representation of elliptic type of $GL_2(F_{\overline{v}}^+\times F_v^+)$ such that

$$\phi^0_R(s_R)\otimes_D\phi^0_L(s_L)=\sum_n(\lambda(n_N^2,b_N^2=0)q^n_{/\mathbb{Q}\,R}\otimes_D\sum_n(\lambda(n_N^2,b_N^2=0))q^n_{/\mathbb{Q}\,L}$$

be defined with $\lambda(n_N^2,b_N^2=0)=n_N=n\cdot N$ as the square root of the eigenvalue n_N^2 of the coset representative

$$\begin{aligned}g_2(n_N^2,0)&=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]\begin{pmatrix}1&0\\0&n_N^2\end{pmatrix}\\&=I\cdot\alpha_{n_N^2}\ .\end{aligned}$$

Suppose that $w^0_{v_{n_{R\times L}}}$ denotes the action of the decomposition group $D_{n^2_N}$ on the irreducible supercuspidal subrepresentation $\Pi_R(n)\otimes_D\Pi_L(n)$.

$\omega^0_{v_{n_{R\times L}}}$ then corresponds to the injective (nilpotent) morphism:

$$\omega^0_{v_{n_{R\times L}}} : \quad \Pi_R(n)\otimes_D\Pi_L(n)\longrightarrow m_R(n)\ \Pi_R(n)\otimes_D m_L(n)\ \Pi_L(n)$$

which generates the multiplicity of the supercuspidal subrepresentation

$$\Pi_R(n)\otimes_D\Pi_L(n)\equiv\lambda(n^2_N,b^2_N=0)\cdot q^n/Q_R\otimes_D\lambda(n^2_N,b^2_N=0)\cdot q^n/Q_L\ .$$

We thus have a family $\omega^0_{v_{R\times L}}$ of injective morphisms at all quadratic places $v_{R\times L}\equiv\overline{v}\times v$ such that:

$$\Pi^0_{R\times L}(\omega^0_{v_{R\times L}}): \quad \mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v)\longrightarrow\phi_R(s_R)\otimes_D\phi_L(s_L)$$

where $\phi_R(s_R)\otimes_D\phi_L(s_L)$ is the global elliptic A_{R-L}-bisemimodule:

$$\phi_R(s_R)\otimes_D\phi_L(s_L)=\sum_{n,b}\lambda_+(n^2_N,b^2_N)\ q^n_{/\mathbb{Q}\,R}\otimes_D\sum_{n,b}\lambda_-(n^2_N,b^2_N)\ q^n_{/\mathbb{Q}\,L}\ .$$

2.23 Proposition

There exists a family $\omega^0_{v_{R\times L}}$ of injective morphisms associated with the action of the set $\{D_{n^2_N}\}_{n^2}$ of decomposition groups such that:

$$\omega^0_{v_{R\times L}} : \quad \Pi^0_{R\times L}\longrightarrow\Pi^0_{R\times L}(\omega^0_{v_{R\times L}})$$

implies the commutative diagram:

$$\begin{tikzcd}
\mathrm{GL}_2(F^+_{\overline{v}}\times F^+_v) \arrow[r, "\Pi^0_{R\times L}"] \arrow[dr, "\Pi^0_{R\times L}(\omega^0_{v_{R\times L}})"'] & \phi^0_R(s_R)\otimes_D\phi^0_L(s_L) \arrow[d, "\omega^0_{v_{R\times L}}"] \\
 & \phi_R(s_R)\otimes_D\phi_L(s_L)
\end{tikzcd}$$

2.24 Corollary

The family of injective morphisms $\omega^0_{v_{R\times L}}$ corresponds to the action of the product $\mathcal{W}_R\times\mathcal{W}_L$ of the Weyl groups acting on the supercuspidal representation of elliptic type $\phi^0_R(s_R)\otimes_D\phi^0_L(s_L)$.

Proof: indeed, $\phi^0_R(s_R)\otimes_D\phi^0_L(s_L)$ consists of the sum of products of pairs of maximal semitori $T^1_R[n]\times T^1_L[n]$ at all quadratic places $v_{R\times L}\equiv\overline{v}\times v$.
And, on the other hand, the action of the set $\{D_{n^2_N}\}_{n^2}$ of the decomposition groups, associated with the family of injective morphisms $\omega^0_{v_{R\times L}}$, corresponds to the action of the product $\mathcal{W}_R\times\mathcal{W}_L$ of the Weyl groups since, for every product $\mathcal{W}_{\overline{v}_n}\times\mathcal{W}_{v_n}\in\mathcal{W}_R\times\mathcal{W}_L$ of Weyl subgroups restricted to the quadratic place $\overline{v}_n\times v_n$, we have:

$$\mathcal{W}_{\overline{v}_n}\times\mathcal{W}_{v_n}=\frac{N_R}{T^1_R[n]}\times\frac{N_L}{T^1_L[n]}$$

where N_L (resp. N_R) is the normalizer of $T^1_L[n]$ (resp. $T^1_R[n]$) in $\Pi_L(n)$ (resp. $\Pi_R(n)$). ∎

3 Applications: The treatment of some conjectures

Chapter 2 deals with the representation of a complex algebraic bilinear semigroup $\mathrm{GL}_2(F_{\overline{\omega}}\times F_\omega)$ into a $B_{F_{\overline{\omega}}}\otimes B_{F_\omega}$-bisemimodule $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$ decomposing into direct sum of bisubsemimodules $M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes M_{F_{\omega_{n,m_{\omega_n}}}}$ characterized by increasing ranks.

A toroidal compactification of the Borel-Serre type of $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$:

1. maps each subsemimodule $M_{F_{\omega_{n,m_{\omega_n}}}}$ (resp. $M_{F_{\overline{\omega}_{n,m_{\omega_n}}}}$) into a two-dimensional semitorus;
2. is such that its boundary is the restricted complex $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$-bisemimodule

$$M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}\otimes M^{\mathrm{res}}_{F^T_{\omega_\oplus}}=\bigoplus_{n=1}^{s}\left(M^{\mathrm{res}}_{F^T_{\overline{\omega}_n}}\otimes M^{\mathrm{res}}_{F^T_{\omega_n}}\right)$$

   covered by the real $\mathrm{GL}_2(F^{+,T}_{\overline{v}}\times F^{+,T}_v)$-bisemimodule

$$M_{F^{+,T}_{\overline{v}_\oplus}}\otimes M_{F^{+,T}_{v_\oplus}}=\bigoplus_{n,m_n}\left(M_{F^{+,T}_{\overline{v}_{n,m_n}}}\otimes M_{F^{+,T}_{v_{n,m_n}}}\right)$$

   at the conditions of proposition 2.5.

The analytic development of the $T_2(F^T_\omega)$-semimodule $M^{\mathrm{res}}_{F^T_{\omega_\oplus}}$ (resp. $T^t_2(F^T_{\overline{\omega}})$-semimodule $M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}$) is the equivalent of the Eisenstein series $\mathrm{EIS}_L(2,n)$ (resp. $\mathrm{EIS}_R(2,n)$) which is a modular form $f_L(z)$ (resp. $f_R(z)$), restricted to the upper (resp. lower) half plane.

So, a modular form on the representation space of the complex bilinear algebraic semigroup $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_\omega)$ is a modular biform $f_R(z)\otimes f_L(z)\in C^\infty(\mathrm{GL}_2(F^T_R\times F^T_L)/\mathrm{GL}_2(\mathbb{Z}/N\,\mathbb{Z})^2)$.

The analytic development of the $T_2(F_v^{+,T})$-semimodule $M_{F_{v_\oplus}^{+,T}}$ (resp. $T_2^t(F_{\overline{v}}^{+,T})$-semimodule $M_{F_{\overline{v}_\oplus}^{+,T}}$) is given by the global elliptic A_L-semimodule $\phi_L(s_L) = \sum\limits_{n,m_n} \lambda_-(n_N^2, m_n^2)\ e^{2\pi inx}$ (resp. A_R-semimodule $\phi_R(s_R) = \sum\limits_{n,m_n} \lambda_+(n_N^2, m_n^2)\ e^{-2\pi inx}$), in such a way that the m_n representatives of the n-th class of $\phi_L(s_L)$ (resp. $\phi_R(s_R)$), which are semicircles, cover the n-th representative of $\mathrm{EIS}_L(2,n)$ (resp. $\mathrm{EIS}_R(2,n)$), which is a $T_L^2[n]$ (resp. $T_R^2[n]$) semitorus.

So, the modular representation of $f_L(z)$ (resp. $f_R(z)$) can be given by a set of n , $1 \le n \le \infty$, two-dimensional semitori $T_L^2[n]$ (resp. $T_R^2[n]$), restricted to the upper (resp. lower) half plane and covered each one by m_n semicircles of the n-th class of the global elliptic semimodule $\phi_L(s_L)$ (resp. $\phi_R(s_R)$).

This kind of modular representation will be used in this chapter in order to analyze the three following conjectures.

3.1 The Shimura-Taniyama-Weil conjecture

The Shimura-Taniyama-Weil conjecture deals with the modular representation of an elliptic curve. Let us recall the main tools of this modular representation.

Let $T_H(N)_R \otimes T_H(N)_L$ be the ring of products of Hecke operators $T_{q_R} \otimes T_{q_L}$ and $U_{q_R} \otimes U_{q_L}$. Assume that a global elliptic A_{R-L}-bisemimodule $\phi_R(s_R) \otimes_D \phi_L(s_L) = \sum\limits_{n,m} (\lambda_+(n_N^2, m_N^2)) q^n_{/\mathbb{Q}\,R} \otimes_D \sum\limits_{n,m} (\lambda_-(n_N^2, m_N^2)) q^n_{/\mathbb{Q}\,L}$ corresponds to each normalized eigenform $f_R(z) \otimes_D f_L(z) = \sum\limits_n a_{n,R} q_R^n \otimes_D \sum\limits_n a_{n,L} q_L^n$, $q_{R,L} = e^{\mp 2\pi iz}$, $z \in \mathbb{C}$, of a product of Hecke operators. $\phi_R(s_R) \otimes_D \phi_L(s_L)$ can then be viewed as an automorphic representation of $f_R(z) \otimes_D f_L(z)$ composed of a double tower of one-dimensional irreducible curves $E_f(n_N, m_N)_{R,L}$: 1) being each one a semitorus $T^1_{R,L}[n, m_n]$ of class n with respect to a quantum $M^I_{[v_1]}$ of class 1 (see section 2.18); 2) having centers and radii given by the pairs $\{\mathrm{cent}(n_N^2, m_n^2), (r(n_N^2, m_n^2))\}$.

3.1.1 Euclidian uniformization of the elliptic curve $E(\mathbb{C})$

Let $E(\mathbb{C})$ be an elliptic curve over $\mathbb{C}$ given by the equation $Y^2 = 4X^3 + AX + B$ ($A, B \in \mathbb{C}$) arising in connection with the nonlinear differential equation $(\wp')^2 = 4\wp^3 + A\wp + B$ where the $\wp$-function is the Weierstrass $\wp$-function which is periodic and related to a lattice in $\mathbb{C}$.

Assume that this lattice in $\mathbb{C}$ is precisely the lattice Λ_ω^2 (resp. $\Lambda_{\overline{\omega}}^2$) in the B_{F_ω}-semimodule M_{F_ω} (resp. $B_{F_{\overline{\omega}}}$-semimodule $M_{F_{\overline{\omega}}}$) as developed in section 2.3.

So, an Euclidian uniformization of our elliptic curve $E(\mathbb{C})$ of the type introduced by B. Mazur in [Ma1] would be obtained by considering the surjective mapping:

$$\mathcal{E}_{\mathrm{GL}_2\to E(\mathbb{C})}: \quad \mathrm{GL}_2(F_R\times F_L)\big/\,\mathrm{GL}_2((\mathbb{Z}/N\cdot\mathbb{Z})^2)\longrightarrow E(\mathbb{C})$$

of the quotient of the complex bilinear algebraic semigroup $\mathrm{GL}_2(F_R\times F_L)$ by the subgroup $\mathrm{GL}_2((\mathbb{Z}/N\cdot\mathbb{Z})^2)$, constituting a representation of the bilattice $\Lambda^2_{\overline{\omega}}\otimes\Lambda^2_{\omega}$. This mapping $\mathcal{E}_{\mathrm{GL}_2\to E(\mathbb{C})}$ identifies the complex points of the elliptic curve $E(\mathbb{C})$ with certain "bipoints" of the orbit space $\mathrm{GL}_2(F_R\times F_L)\big/\,\mathrm{GL}_2((\mathbb{Z}/N\cdot\mathbb{Z})^2)$ with respect to $\Lambda^2_{\overline{\omega}}\otimes\Lambda^2_{\omega}$.

3.1.2 Hyperbolic uniformization of the elliptic curve $E(\mathbb{C})$

The quotient bilinear semigroup $\mathrm{GL}_2(F_R\times F_L)\big/\,\mathrm{GL}_2((\mathbb{Z}/N\cdot\mathbb{Z})^2)$ is a bisemispace whose representation is the $\mathrm{GL}_2(F_{\overline{\omega}}\times F_{\omega})$-bisemimodule $M_{F_{\overline{\omega}}}\otimes M_{F_{\omega}}$.

As it was developed in chapter 2, a modular representation of $M_{F_{\overline{\omega}_\oplus}}\otimes M_{F_{\omega_\oplus}}$ can be only obtained if we consider the restricted $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_{\omega})$-bisemimodule $M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}\otimes M^{\mathrm{res}}_{F^T_{\omega_\oplus}}=\bigoplus\limits_{n=1}^{s}\left(M^{\mathrm{res}}_{F^T_{\overline{\omega}_n}}\otimes M^{\mathrm{res}}_{F^T_{\omega_n}}\right)$, $1\le n\le s\le\infty$, having a cuspidal representation given by the product, right by left, $\mathrm{EIS}_R(2,n)\otimes\mathrm{EIS}_L(2,n)$ of the equivalent of Eisenstein series (see section 2.8).

Let $f_R(z)\otimes_D f_L(z)\simeq\mathrm{EIS}_R(2,n)\otimes_D\mathrm{EIS}_L(2,n)$ be the normalized eigenform of the above mentioned products of Hecke operators: it is a product, right by left, of cuspidal forms expanded in formal power series.

Thus, $f_R(z)\otimes_D f_L(z)$ constitutes a cuspidal representation of the restricted $\mathrm{GL}_2(F^T_{\overline{\omega}}\times F^T_{\omega}))$-bisemimodule $M^{\mathrm{res}}_{F^T_{\overline{\omega}}}\otimes_D M^{\mathrm{res}}_{F^T_{\omega}}$.

A hyperbolic uniformization of our elliptic curve $E(\mathbb{C})$ will be reached by considering the surjective mapping:

$$\mathcal{H}_{\mathrm{GL}^{\mathrm{res}}_{2_{\mathbb{C}}}\to E(\mathbb{C})}: \quad M^{\mathrm{res}}_{F^T_{\overline{\omega}}}\otimes_D M^{\mathrm{res}}_{F^T_{\omega}}\longrightarrow E(\mathbb{C})$$

identifying the complex points of $E(\mathbb{C})$ with certain complex bipoints of $M^{\mathrm{res}}_{F^T_{\overline{\omega}}}\otimes_D M^{\mathrm{res}}_{F^T_{\omega}}$.

So, the hyperbolic uniformization of $E(\mathbb{C})$ is especially obtained by getting rid of the multiples of the subbisemimodules $M_{F^T_{\overline{\omega}_n}}\otimes M_{F^T_{\omega_n}}$, $1\le n\le s\le\infty$, i.e. by considering:

$$\begin{aligned}
M^{\mathrm{res}}_{F^T_{\overline{\omega}_\oplus}}\otimes_D M^{\mathrm{res}}_{F^T_{\omega_\oplus}} &= \sum_{n=1}^{s}\sum_{m_{\omega_n}}\left(M_{F^T_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes_D M_{F^T_{\omega_{n,m_{\omega_n}}}}\right)-\sum_{n=1}^{s}\sum_{m_{\omega_n}>1}\left(M_{F^T_{\overline{\omega}_{n,m_{\omega_n}}}}\otimes_D M_{F^T_{\omega_{n,m_{\omega_n}}}}\right)\\
&= \sum_{n=1}^{s}\left(M^{\mathrm{res}}_{F^T_{\overline{\omega}_n}}\otimes_D M^{\mathrm{res}}_{F^T_{\omega_n}}\right)\ .
\end{aligned}$$

As $f_R(z)\otimes_D f_L(z)$ is a cuspidal representation of $M^{\rm res}_{F^T_{\overline{\omega}}}\otimes_D M^{\rm res}_{F^T_{\omega}}$, it constitutes a modular representation of the elliptic curve by the surjective mapping:

$$\mathcal{H}^{\rm cusp}_{{\rm GL}^{\rm res}_{2_{\mathbb{C}}}\to E(\mathbb{C})}: \quad f_R(z)\otimes_D f_L(z) \longrightarrow E(\mathbb{C})$$

if we take into account the mapping $\mathcal{H}_{{\rm GL}^{\rm res}_{2_{\mathbb{C}}}\to E(\mathbb{C})}$.

This hyperbolic uniformization [Rib2] is realized in the upper and in the lower half planes which are periodic respectively with the congruence subgroups $\Gamma_L(N)$ and $\Gamma_R(N)$ (representations respectively of the Hecke operators U_{q_L} and U_{q_R}) being restricted subgroups of $\Gamma(N)$ (or $\Gamma_1(N)$ or $\Gamma_0(N)$) and $\Gamma^t(N)$ (or $\Gamma^t_1(N)$ or $\Gamma^t_0(N)$) respectively (see section 2.12).

3.1.3 Hyperbolic uniformization of arithmetic type of the elliptic curve $E(\mathbb{Q})$

The next step consists in envisaging the modular representation of the elliptic curve $E(\mathbb{Q})$, covering its complex equivalent $E(\mathbb{C})$, by means of the global elliptic A_{R-L}-bisemimodule $\phi_R(s_R)\otimes_D\phi_L(s_L)$. So, we have to consider the diagram:

$$\begin{array}{ccc}
f_R(z)_{\rm res}\otimes_D f_L(z)_{\rm res} & \xrightarrow{\mathcal{H}^{\rm cusp\ (res)}_{{\rm GL}^{\rm res}_{2_{\mathbb{C}}}\to\mathbb{C}}} & E(\mathbb{C}) \\
{\scriptstyle\mathcal{M}_{\phi\to f}}\Big\uparrow & & \Big\uparrow{\scriptstyle\mathcal{M}_{E(\mathbb{Q})\to E(\mathbb{C})}} \\
\phi_R(s_R)_{\rm res}\otimes_D \phi_R(s_L)_{\rm res} & \xrightarrow{\mathcal{H}^{\rm cusp\ (res)}_{{\rm GL}^{\rm res}_{2_{\mathbb{C}}}\to\mathbb{Q}}} & E(\mathbb{Q})
\end{array}$$

in such a way that the covering $\mathcal{M}_{E(\mathbb{Q})\to E(\mathbb{C})}$ of the elliptic curve $E(\mathbb{C})$ over $\mathbb{C}$ by the elliptic curve $E(\mathbb{Q})$ over $\mathbb{Q}$ can be reached throughout the covering $\mathcal{M}_{\phi\to f}$ of the restricted cuspidal biform $f_R(z)_{\rm res}\otimes_D f_L(z)_{\rm res}$ by the restricted global elliptic bisemimodule $\phi_R(s_R)_{\rm res}\otimes_D\phi_R(s_L)_{\rm res}$, as it will be developed in the following.

As the mapping $\mathcal{M}_{\phi\to f}$ was justified in chapter 2, the covering mapping $\mathcal{M}_{E(\mathbb{Q})\to E(\mathbb{C})}$ of the elliptic curve $E(\mathbb{C})$ by the elliptic curve $E(\mathbb{Q})$ will be justified if the surjective mapping $\mathcal{H}^{\rm cusp(res)}_{{\rm GL}^{\rm res}_{2_{\mathbb{R}}}\to E(\mathbb{Q})}$ of the elliptic curve $E(\mathbb{Q})$ by the global elliptic bisemimodule $\phi_R(s_R)_{\rm res}\otimes_D \phi_R(s_L)_{\rm res}$ can be justified. This surjective mapping $\mathcal{H}^{\rm cusp(res)}_{{\rm GL}^{\rm res}_{2_{\mathbb{R}}}\to E(\mathbb{Q})}$ corresponds precisely to the hyperbolic uniformization of arithmetic type of the elliptic curve $E(\mathbb{Q})$ because it corresponds to a modular representation of the elliptic curve $E(\mathbb{Q})$ by $(\phi_R(s_R)_{\rm res}\otimes_D \phi_R(s_L)_{\rm res})$.

Now, describing the elliptic curve $E(\mathbb{Q})$ globally over $\mathbb{Q}$ is equivalent to consider the set $\{E(\mathbb{Z}/n\,\mathbb{Z})\}_{n\in\mathbb{N}}$ of the elliptic curves, and, especially, the set $\{E(\mathbb{F}_p)\}_p$ of elliptic curves over $\mathbb{F}_p$, p being a prime number (because our equation of $E(\mathbb{Q})$, having integer coefficients, can be reduced modulo p) [Tat], [Ma2].

As mentioned above, $\phi_R(s_R)\otimes_D\phi_R(s_L)$ is the direct sum of the n sets of products, right by left,

$$\{E_f(n_N,m_n)_R)\otimes E_f(n_N,m_n)_L\}_{m_n}\equiv\{T^1_R(n,m_n]\otimes T^1_M[n,m_n]\}_{m_n}$$

of one-dimensional irreducible curves $E_f(n_N,m_N)$ which are semitori $T^1[n,m_n]$.

So, at each integer n , we have a subset of m_n , $m_n\in\mathbb{N}$, products of semitori $T^1_L[n,m_n]$, localized in the upper half plane, by their equivalents $T^1_R[n,m_n]$ in the lower half plane in such a way that $T^1_R[n,m_n]$ be, for instance, projected on $T^1_L[n,m_n]$.

These m_n-semitori $T^1_L[n,m_n]$ and $T^1_R[n,m_n]$, generated under the action of the decomposition group $D_{n^2_N}$ according to section 2.13, are isomorphic.

Taking into account that:

a) $E(\mathbb{F}_p)\simeq(\mathbb{Z}/p\,\mathbb{Z})^2$,

b) $T^1_L(p,m_p]$ (resp. $T^1_R[p,m_p]$) is a cyclic semigroup whose order is a multiple of p ,

it is reasonable to associate with an elliptic curve $E(\mathbb{F}_p)$ (having good reduction at p), the subset $\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}$ of m_p products of semitori, p being a prime integer, in such a way that there exists a natural surjective mapping:

$$\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}\longrightarrow E(\mathbb{F}_p)$$

which identifies the elliptic curve $E(\mathbb{F}_p)$ with the orbit space of the p-th class of products, right by left, of semitori of the global elliptic bisemimodule $\phi_R(s_R)\otimes_D\phi_L(s_L)$ under the action of the decomposition group $D_{p^2_N}$.

3.1.4 Proposition

If we have the natural surjective mapping:

$$\{E_f(p_N,m_p)_R)\otimes E_f(p_N,m_p)_L\}_{m_p}\longrightarrow E(\mathbb{F}_p)\ ,$$

then the elliptic curve $E(\mathbb{F}_p)$ will have m_p generating points given by:

$$\begin{aligned}\#E(\mathbb{F}_p)&=(|\det\rho_{\lambda(p^2_N,m^2_p)}-\operatorname{trace}\rho_{\lambda(p^2_N,m^2_p)}+1|)^{\frac12}\\&=m_p\end{aligned}$$

where $\det\rho_{\lambda(p^2_N,m^2_p)}$ *and* $\operatorname{trace}\rho_{\lambda(p^2_N,m^2_p)}$ *are given in proposition 2.15.*

Proof.

1. The sum of the m_p products, right by left, of semitori is given, according to section 2.20, by:

$$\bigoplus_{m_p} (E_f(p_N,m_p)_R) \otimes E_f(p_N,m_p)_L) = \bigoplus_{m_p} \left(\lambda_+(p_N^2,m_p^2)\ e^{-2\pi ipx} \otimes_D \lambda_+(p_N^2,m_p^2)\ e^{2\pi ipx}\right)\ ,$$

   where the coefficients $\lambda_\pm(p_N^2,m_p^2)$ are eigenvalues of products of Hecke operators given by the coset representatives $g_2(p_N^2,b_N)$ in sections 2.12, 2.13, and in proposition 2.15.

2. Recall that the number of points on an elliptic curve $E(\mathbb{F}_p)$ is traditionally given by:

$$\#E(\mathbb{F}_p) = p+1-a_p$$

   where a_p results from $T_p(f) = a_p\ f$ if

   - f is an eigenfunction of the coset representative T_p of the Hecke operator;
   - a_p is the sum of the eigenvalues of T_p.

   A similar formula can be introduced by taking into account the present context.

3. As the number of Galois automorphisms on a semitoric curve $E_f(p_N,m_p)$ is equal to its rank $r_{E_f(p_N,m_p)} = p_N = p\cdot N$, it is reasonable to associate with the orbit space of the subset $\{E_f(p_N,m_p)_R \otimes E_f(p_N,m_p)_L\}_{m_p}$ the elliptic curve $E(\mathbb{F}_p)$ over $\mathbb{F}_p$ and to think that the number of generating points on $E(\mathbb{F}_p)$ could be given by the rank $r^0_{E_f^2(p_N,m_p)} = p_N^2$ of this orbit space [Ma1] viewed as a product, right by left, of semicircles. Now, this rank $r^0_{E_f^2(p_N,m_p)}$ is precisely given by

$$\begin{aligned}\det \rho_{\lambda(p_N^2,m_p^2)} &= \lambda_+(p_N^2,m_p^2) \centerdot \lambda_-(p_N^2,m_p^2)\\ &= p_N^2\end{aligned}$$

   according to proposition 2.15.

   So, $\#E(\mathbb{F}_p)$ could be given by:

$$\#E(\mathbb{F}_p) = (|\det \rho_{\lambda(p_N^2,nm_p^2)}|)^{\frac{1}{2}}\ .$$

4. But, the orbit space of these toric curves $E_f(p_N,m_p)$ must be centered at the origin in order to have an isogeny [Kna].

Then, we have that:

$$\begin{aligned}\#E(\mathbb{F}_p) &= \Big(|\det\rho_{\lambda(p_N^2,m_p^2)} - \operatorname{trace}\rho_{\lambda(p_N^2,m_p^2)} + 1|\Big)^{\frac{1}{2}}\\ &= \big(|p_N^2 - (1+m_p^2+p_N^2)+1|\big)^{\frac{1}{2}} = m_p\end{aligned}$$

where:

- $\operatorname{trace}\rho_{\lambda(p_N^2,m_p^2)} = 1+m_p^2+p_N^2$, according to proposition 2.15, contributes to the above mentioned isogeny;

- the term " $+1$ " comes from the point at infinity. ∎

3.1.5 Modular representation of the elliptic curve $E(\mathbb{Q})$

The problem which arises now is to precise the set $\{E(\mathbb{F}_p)\}_p$ of elliptic curves $E(\mathbb{F}_p)$ over $\mathbb{F}_p$ (p varying) which are locally equivalent to our elliptic curve $E(\mathbb{Q})$ to which a modular representation must correspond. So, we have to introduce the Hecke L-series of the elliptic curve $E(\mathbb{Q})$ and, more generally, the Hecke L-series associated to the cusp forms $f_L(z)$ and $f_R(z)$.

3.1.6 Definition: Hecke L-series

Let $\lambda_+(q_N^2,m_N^2)$ and $\lambda_-(q_N^2,m_N^2)$ be the eigenvalues of the Hecke operators $U_{q_R}\otimes U_{q_L}(q\mid N)$ and $T_{q_R}\otimes T_{q_L}(q\nmid N)$ (see section 2.3).

We can then define the Hecke L-series:

$$L_R(s_-) = \sum_n \lambda_\mp(n_N^2,m_N^2)n^{-s_-}\ , \qquad s_-\in\mathbb{C} \quad \text{with } \{\operatorname{Im} s_- < 0\}\ ,$$

in the lower half plane, and

$$L_L(s_+) = \sum_n \lambda_\mp(n_N^2,m_N^2)n^{-s_+}\ , \qquad s_+\in\mathbb{C} \quad \text{with } \{\operatorname{Im} s_+ > 0\}$$

in the upper half plane, with respect to

$$\lambda_\mp(n_N^2,m_N^2) = \frac{(1+m_N^2+n_N^2)\mp[(1+m_N^2+n_N^2)^2-4n_N^2]^{\frac{1}{2}}}{2}$$

(see section 2.13 and proposition 2.15), and associated respectively with a pseudo-ramified right (resp. left) cusp form

$$f_R(z) = \sum_n \lambda_\mp(n_N^2,m_N^2)\ e^{2\pi i n z} \quad (\text{resp.} \quad f_L(z) = \sum_n \lambda_\mp(n_N^2,m_N^2)\ e^{-2\pi i n z}\ ); \quad z\in\mathbb{C}\ .$$

3.1.7 Proposition

Let $L_R(s_-)$ and $L_L(s_+)$ be the Hecke L-series defined respectively in the lower and in the upper half plane. The $L^{\deg}_{R-L}(\mathrm{Re}(s))$ Hecke L-series can be defined from their product (degenerate case):

$$L^{\deg}_{R-L}(\mathrm{Re}(s)) = L_R(s_-)\cdot L_L(s_+) = \sum_n \lambda^2_{\pm}(n^2_N,m^2_N)n^{-2x}\ , \qquad x\in \mathrm{Re}(s_{\pm})$$

which corresponds to the classical Eulerian development [La]*:*

$$L^{\deg}_{R-L}(\mathrm{Re}(s)) = \prod_{q\mid N}(1-\lambda^2_{\mp}(q^2_N)q^{-2x})^{-1}\prod_{q\nmid N}(1-\lambda^2_{\mp}(q^2_N)\ \varepsilon(q)^2q^{2k-2x-2})^{-1}$$

where

- $\varepsilon(q):(\mathbb{Z}\ /\ N\mathbb{Z})^*\to\mathbb{C}^*$ *is the Dirichlet character*

- $\lambda^2_{\mp}(q^2_N)=\sum_m \lambda^2_{\mp}(q^2_N,m^2_N)$.

Proof.

a) As the Hecke L-series $L_{R-L}(\mathrm{Re}(s))$ is defined on a space of semisimple type, the off-diagonal terms of the product of $L_R(s_-)$ by $L_L(s_+)$ must be zero.

b) Taking into account that the $\lambda_-(q^2_N,m^2_N)$ are eigenvalues of $(D_{q^2_N;m^2_N})\cdot\alpha_{q^2_N}$ (see section 2.13) and considering the statement of proposition 2.15, it is clear that $L^{\deg}_{R-L}(\mathrm{Re}(s))$ is also an Artin L-function. ∎

3.1.8 Corollary

The Hecke L-series $L_{R-L}(s)=L_R(s_-)\cdot L_L(s_+)$ have also the following Eulerian development (nondegenerate case):

$$L_{R-L}(s) = \prod_{q\mid N}(1-\lambda_{\mp}(q^2)q^{-s_-})^{-1}\prod_{q\nmid N}(1-\lambda_{\mp}(q^2_N)\ \varepsilon(q)q^{k-s_--1})^{-1}$$
$$\cdot\prod_{q\mid N}(1-\lambda_{\mp}(q^2_N)q^{-s_+})^{-1}\prod_{q\nmid N}(1-\lambda_{\mp}(q^2_N)\ \varepsilon(q)q^{k-s_+-1})^{-1}\ .$$

3.1.9 Hecke L-series of the elliptic curve $E(\mathbb{Q})$

Considering the canonical diagonal injective mapping of $\mathbb{Q}$ into the adele ring $\mathbb{A}$, which can be seen from the discrete topology from $\mathbb{A}$ into $\mathbb{Q}$, [Rob], we can admit that the injection

$$E_{\mathbb{Q}\to\prod_q} : \quad E(\mathbb{Q}) \longrightarrow E(\prod_q \mathbb{F}_q) ,$$

where $\prod_q$ is taken over all primes q , allows to transfer the study of $E(\mathbb{Q})$ onto a subset of $E(\prod_q \mathbb{F}_q)$.

But, if we realize that the kernel $\mathrm{Ker}(E^{-1}_{\mathbb{Q}\to\prod_q})$ of the map $E^{-1}_{\mathbb{Q}\to\prod_q}$ refers precisely to these primes " $p_\perp$ " which do not enter in the generation of the elliptic curve $E(\mathbb{Q})$, then the Hecke L-series $L_{R,L}(s_\mp)$ can be partitioned into two complementary subseries following:

$$L_{R,L}(s_\mp) = L_{R,L}(s_\mp, E(\mathbb{Q})) + L_{R,L}(s_\mp, E(\prod_{p_\perp} \mathbb{F}_{p_\perp}))$$

where

- $L_{R,L}(s_\mp, E(\mathbb{Q}))$ is the L-subseries of the elliptic curve $E(\mathbb{Q})$ such that their Euler factors refer to the set of primes $\{p\}$ complementary to the set of primes $p_\perp$:

$$L_{R,L}(s_\mp, E(\mathbb{Q})) = \sum_{n_g} \lambda_\mp(n^2_{g_N}, m^2_N) n_g^{-s_\mp} , \quad n_g \le n .$$

- $L_{R,L}(s_\mp, E(\prod_{p_\perp} \mathbb{F}_{p_\perp}))$ is the "virtual" subseries of $L_{R,L}(s_\mp)$ with regard to $L_{R,L}(s_\mp, E(\mathbb{Q}))$;

- $L_{R,L}(s_\mp, E(\prod_{p_\perp}(\mathbb{F}_{p_\perp}))) = \sum_{n-n_g} \lambda_\mp((n_N - n_{g_N})^2, m^2_N)\ (n-n_g)^{-s_\mp}$, $(n_N - n_{g_N}) \in I\!N$, $n_N \equiv n \cdot N$, $n \ge n_g$;

- $\{q\} = \{p^\perp\} \cup \{p\}$.

3.1.10 Proposition (Hyperbolic uniformization of arithmetic type)

Let

$$
\begin{aligned}
& L_R(s_-,E(\mathbb{Q}\,))\otimes_D L_L(s_+,E(\mathbb{Q}\,)) \\
&\quad = \textstyle\sum\limits_{n_g}(\lambda_-(n^2_{g_N},m^2_{n_g})\; n_g^{-s_-}\otimes_D \lambda_+(n^2_{g_N},m^2_{n_g})\; n_g^{-s_+}) \\
&\quad = \textstyle\prod\limits_{p\mid N}(1-\lambda_\mp(p^2_N,m^2_p)\; p^{-s_-})^{-1} \;\; \textstyle\prod\limits_{p\nmid N}(1-\lambda_\mp(p^2_N,m^2_p)\;\varepsilon(p)\; p^{k-s_--1})^{-1} \\
&\qquad \textstyle\prod\limits_{p\mid N}(1-\lambda_\mp(p^2_N,m^2_p)\; p^{-s_+})^{-1} \;\; \textstyle\prod\limits_{p\nmid N}(1-\lambda_\mp(p^2_N,m^2_p)\;\varepsilon(p)\; p^{k-s_+-1})^{-1}
\end{aligned}
$$

be the restricted product of L-subseries attached to the elliptic curve $E(\mathbb{Q}\,)$ and corresponding to the restricted cuspidal biform of weight two and level N :

$$
\begin{aligned}
& f_R(z)_{\rm res}\otimes_D f_L(z)_{\rm res} \\
&\quad = \textstyle\sum\limits_{n_g}\left(\lambda_-(n^2_{g_N},m^2_{n_g})\; e^{-2\pi i n_g z}\otimes_D \lambda_+(n^2_{g_N},m^2_{n_g})\; e^{2\pi i n_g z}\right)\;, \quad z\in\mathbb{C}\;,
\end{aligned}
$$

covered by the restricted global elliptic bisemimodule:

$$
\begin{aligned}
& \phi_R(s_R)_{\rm res}\otimes_D \phi_L(s_L)_{\rm res} \\
&\quad = \textstyle\sum\limits_{n_g}\sum\limits_{m_{n_g}}\left(\lambda_-(n^2_{g_N},m^2_{n_g})\; e^{-2\pi i n_g x}\otimes_D \lambda_+(n^2_{g_N},m^2_{n_g})\; e^{2\pi i n_g x}\right)\;, \quad x\in\mathbb{R}\;.
\end{aligned}
$$

Then, the modular representation of the elliptic curve $E(\mathbb{Q}\,)$ over $\mathbb{Q}$ will be worked out from the " p " sets of surjective mappings:

$$\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}\longrightarrow E(\mathbb{F}_p)\;,$$

for all prime p entering into the restricted eulerian product of $L_R(s_-,E(\mathbb{Q}\,))\otimes_D L_L(s_+,E(\mathbb{Q}\,))$, in such a way that the orbit spaces of $\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}_{m_p}$ are associated in the sense of proposition 3.1.4, with the elliptic curves $E(\mathbb{F}_p)$, for the above mentioned p primes, having good reduction and conductor N . This modular representation of the elliptic curve $E(\mathbb{Q}\,)$ corresponds to its hyperbolic uniformization of arithmetic type [Ma1]*, and, thus to the Shimura-Taniyama-Weil conjecture.*

3.1.11 Connection with Diophantine equations

As it is well known, the Shimura-Taniyama-Weil conjecture is related to the problem of Diophantine equations especially by means of the Mordell-Weil group of the elliptic curve $E(\mathbb{Q}\,)$. Indeed, Mordell proved that $E(\mathbb{Q}\,)$ is a finitely generated abelian group, the generators of $E(\mathbb{Q}\,)$ being the rational points on the curve or the rational solutions

of the equation [Gou]. So, in the context developed here, the generators of $E(\mathbb{Q})$ are precisely the p sets of m_p points associated with the p surjective mappings $\{E_f(p_N,m_p)_R\otimes E_f(p_N,m_p)_L\}\longrightarrow E(\mathbb{F}_p)$ introduced in proposition 3.1.4.

3.2 The Riemann conjecture

3.2.1 The trivial zeros and the zeta functions

The "(pseudo-)unramified" Hecke L-series (with $N=1$), correspond to the classical L-series: they are in one-to-one correspondence with the classical zeta function $\zeta(s)$, are denoted $L^{nr}_{R,L}(s_\mp)$ and given by:

$$\begin{aligned}
L^{nr}_R(s_-)&=\sum_n\lambda^{nr}_\mp(n^2,m^2)\ n^{-s_-}\ ,\\
L^{nr}_L(s_+)&=\sum_n\lambda^{nr}_\mp(n^2,m^2)\ n^{-s_+}\ ,
\end{aligned}$$

where $\lambda^{nr}_\mp(n^2,m^2)$ is introduced in corollary 2.1.6.

The Hamburger theorem [Chan] tells us that $L^{nr}_{R,L}(s_\mp)=a_1\zeta_{R,L}(s_\mp)$ where $a_1\in\mathbb{C}$ and where $\zeta_{R,L}(s_\mp)=\sum\limits_n n^{-s_\mp}$ is the classical zeta function associated with the right or the left case.

As it is done classically [Ing], let us introduce the right (resp. left) functions:

$$\begin{aligned}
\xi_{R,L}(s_\mp)&=(s_\mp-1)\Pi^{-\frac12 s_\mp}\Gamma(\tfrac12 s_\mp+1)\zeta_{R,L}(s_\mp)\\
&=h_{R,L}(s_\mp)\zeta_{R,L}(s_\mp)
\end{aligned}$$

which satisfies the functional equation $\xi_{R,L}(1-s_\mp)=\xi_{R,L}(s_\mp)$.

The poles of $h_{R,L}(s_\mp)$, i.e. of the gamma function $\Gamma(\frac12 s_\mp+1)$, are simple ones at $s_\mp=-2,-4,-6,\cdots,-2n$. Since these are points at which $\xi_{R,L}(s_\mp)$ is regular and not zero, they must be simple zeros of $\zeta_{R,L}(s_\mp)$ [Tit].

The trivial zeros $s_\mp=-2,-4,\cdots,-2n$ of $\zeta_{R,L}(s_\mp)$ will be interpreted in the present context as being equal at a sign to $2f_{\overline{v}_n}$ (resp. $2f_{v_n}$) where f_{v_n} is the global class residue degree of the v_n-th real place. The factor 2 proceeds from the fact that the left (resp. right) places are defined in the upper (resp. lower) half space on which are defined curves isomorphic to $1D$-semitori: thus, these $1D$-semitori must be doubled leading to analytic continuation to the whole s-plane and their global class residue degrees must also be multiplied by two.

More concretely, the trivial zeros of $\zeta_{R,L}(s_\mp)$ can be interpreted as follows.

As the (pseudo-)unramified cuspidal biform $f_R^{nr}(z) \otimes_D f_L^{nr}(z)$ is in one-to-one correspondence with the product, right by left, $\zeta_R(s_-) \otimes_D \zeta_L(s_+) = \sum\limits_n n^{-2x}$, $x \in \mathrm{Re}(s_\mp)$, of classical zeta functions, generating a zeta function $\sum\limits_n n^{-2x}$ over $\mathbb{R}$, it appears that the supercuspidal representation of the cuspidal biform $f_R^{nr}(z) \otimes_D f_L^{nr}(z)$ may degenerate into the pseudo-unramified simple real global elliptic bisemimodule:

$$\phi_R^{nr,0}(s_R) \otimes_D \phi_L^{nr,0}(s_L) = \sum_n (\lambda^{nr}(n^2)\ e^{-2\pi inx} \otimes_D \lambda^{nr}(n^2)\ e^{2\pi inx})\ ,$$

the (pseudo-)unramification leading to consider the value $N = 1$ for the conductor and the "simplicity" consisting in considering the irreducible curves $\pi_L(n)$ and $\pi_R(n)$ of global class n having multiplicity one (see sections 2.20 and 2.22): so $m_R(n) = m_L(n) = 1$.

This global elliptic bisemimodule $\phi_R^{nr,0}(s_R) \otimes_D \phi_L^{nr,0}(s_L)$ can be interpreted geometrically as the direct sum of products of irreducible (pseudo-)unramified semitoric curves corresponding to each other by pairs of same level n and constituting the automorphic representation space of the pseudo-unramified algebraic bilinear semigroup $\mathrm{GL}_2(F_{\overline{v}}^{+,T,nr} \times F_v^{+,T,nr})$ and the orbit space of $f_R^{nr}(z) \otimes_D f_L^{nr}(z)$.

The interpretation of the trivial zeros of $\zeta_{R,L}(s_\mp)$ can be given by means of the homomorphism $H_{\phi_{R\times L}\to\zeta_{R\times L}}$ introduced in the following proposition.

3.2.2 Proposition

The one-to-one correspondence between the double pseudo-unramified simple global elliptic bisemimodule $2\phi_R^{nr,0}(s_R) \otimes_D 2\phi_L^{nr,0}(s_L)$ *and the product, right by left, of zeta functions* $\zeta_R(s_-) \otimes_D \zeta_L(s_+)$ *is given by the homomophism:*

$$\begin{aligned} H_{\phi_{R\times L}\to\zeta_{R\times L}} : \quad 2\phi_R^{nr,0}(s_R) \otimes_D 2\phi_L^{nr,0}(s_L) &= \sum_n (2\lambda^{nr}(n^2)e^{-2\pi inx} \otimes_D 2\lambda^{nr}(n^2)e^{2\pi inx} \\ &\longrightarrow \zeta_R(s_-) \otimes_D \zeta_L(s_+) = \sum_n (n^{-s_-} \otimes_D n^{-s_+}) \end{aligned}$$

such that the kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ *of* $H_{\phi_{R\times L}\to\zeta_{R\times L}}$ *maps into the set of products, right by left, of the trivial zeros of* $\zeta_R(s_-)$ *and of* $\zeta_L(s_+)$ *.*

Proof. Indeed, the kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ is given by the set of bipoints:

$$\{\sigma_{n_R} \times \sigma_{n_L} = 4(\lambda^{nr})^2(n^2) = 2\lambda^{nr}(n^2)(e^{-2\pi inx} \mid x=0) \times 2\lambda^{nr}(n^2)(e^{2\pi inx} \mid x=0))\}_{n=1}^{\infty}$$

localized on the real axis σ in such a way that:

- the left (resp. right) point $\sigma_{n_L} = 2\lambda^{nr}(n^2)(e^{2\pi inx} \mid x = 0)$ (resp. $\sigma_{n_R} = 2\lambda^{nr}(n^2)(e^{-2\pi inx} \mid x = 0)$) corresponds to the degeneracy of the irreducible (toric) curve $2\lambda^{nr}(n^2)e^{2\pi inx}$ (resp. $2\lambda^{nr}(n^2)e^{-2\pi inx}$);

- the bipoint $\sigma_{n_R}\times\sigma_{n_L}$ is defined as the product of the right point σ_{n_R} by its left equivalent σ_{n_L} , and is equal to $\sigma_{n_R}\times\sigma_{n_L}=4(\lambda^{nr})^2(n^2)=4f^2_{v_n}=4\ n^2$.

So, as the kernel $\mathrm{Ker}(H_{\phi_{R\times L}\to\zeta_{R\times L}})$ of $H_{\phi_{R\times L}\to\zeta_{R\times L}}$ is the set of bipoints $\{\sigma_{n_R}\times\sigma_{n_L}=4\ n^2\}_{n=1}^{\infty}$ corresponding to the trivial zeros of $\zeta_R(s_-)\otimes\zeta_L(s_+)$, we can see that the trivial zeros of $\zeta_{R,L}(s_\mp)$ are the integers $s_\mp=-2,-4,\cdots,-2n$. ∎

The non-trivial zeros of $\zeta_{R,L}(s_\mp)$ can then be generated from the corresponding trivial zeros by taking into account the following considerations.

3.2.3 The non-trivial zeros of the zeta functions

Let

$$\alpha_{4n^2}=\begin{pmatrix}4n^2 & 0\\ 0 & 1\end{pmatrix}$$

be the split Cartan subgroup element associated with the quadratic place $v^2_{2n}=\overline{v}_{2n}\times v_{2n}$.
Let

$$D_{4n^2;i^2}=\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 0\end{pmatrix}$$

be the coset representative of the Lie algebra of the decomposition group acting on α_{4n^2} : it corresponds to the coset representative of an unipotent Lie algebra mapping in the topological Lie algebra $\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$. Let $\lambda^{nr}_{+}(4n^2,i^2)$ and $\lambda^{nr}_{-}(4n^2,i^2)$ be the eigenvalues of $(D_{4n^2,i^2}\cdot\alpha_{4n^2})$ interpreted as weights in section 2.16.

The Lie algebra $(\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$ consists in vector fields on the Lie group $GL_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$. Let

$$\mathcal{E}_{4n^2}=\begin{pmatrix}E_{4n^2} & 0\\ 0 & 1\end{pmatrix}$$

be the infinitesimal generator of this Lie algebra $\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$ corresponding to the quadratic place v^2_{2n} .

Every root of this Lie algebra $\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$ is determined by the (equivalent)

eigenvalues $\lambda^{nr}_{\pm}(4n^2,i^2,E_{4n^2})$ of

$$D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}\cdot\alpha_{4n^2}=\left[\begin{pmatrix}1&i\\0&1\end{pmatrix}\begin{pmatrix}1&0\\i&1\end{pmatrix}\right]\begin{pmatrix}E_{4n^2}&0\\0&1\end{pmatrix}\begin{pmatrix}4n^2&0\\0&1\end{pmatrix}\ .$$

They are given by:

$$\lambda^{nr}_{\pm}(4n^2,i^2,E_{4n^2})=\frac{1\pm i\sqrt{16n^2\cdot E_{4n^2}-1}}{2}\ .$$

The fact that the representation of the Lie algebra $\mathfrak{g}\ell_2(F^{+,T,nr}_{\overline{v}}\times F^{+,T,nr}_{v})$ is composed of a tower of sections of a vertical tangent bundle explains why the coset representative D_{4n^2,i^2} has been chosen with $b=i\equiv\sqrt{-1}$.

$D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}$ is in fact a coset representative of the Lie algebra of the decomposition group $D_{i^2}(\mathbb{Z})_{|4n^2}$ noted $\mathrm{Lie}(D_{i^2}(\mathbb{Z})_{|4n^2})$.

Then, we have the following proposition.

3.2.4 Proposition

Let $D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}$ be a coset representative of the Lie algebra of the decomposition group $D_{i^2}(\mathbb{Z})_{|4n^2}$ and let α_{4n^2} be the corresponding split Cartan subgroup element.
Then, the products of the pairs of the trivial zeros of the Riemann zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$ are mapped into the products of the corresponding pairs of the non-trivial zeros as follows:

$$\begin{aligned}D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}:\quad &\det(\alpha_{4n^2})\longrightarrow\det(D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}\cdot\alpha_{4n^2})_{SS}\\ &\{(-2n)\cdot(-2n)\}\longrightarrow\{\lambda^{nr}_{+}(4n^2,i^2,E_{4n^2})\cdot\lambda^{nr}_{-}(4n^2,i^2,E_{4n^2})\}\ ,\qquad\forall\ n\in\mathbb{N}\ ,\end{aligned}$$

where $(\ )_{SS}$ denotes the semisimple form of $D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}\cdot\alpha_{4n^2}$.

Proof. The squares of the trivial zeros $(-2n)^2$ can be interpreted as the squares of the global class residue degrees.
As $D_{4n^2,i^2}\cdot\mathcal{E}_{4n^2}$ is of Galois type, it maps squares of trivial zeros $(-2n)^2$ into products of corresponding pairs of other zeros $\lambda^{nr}_{+}(4n^2,i^2,E_{4n^2})\cdot\lambda^{nr}_{-}(4n^2,i^2,E_{4n^2})$ which are non-trivial zeros since $\lambda^{nr}_{\pm}(4n^2,i^2,E_{4n^2})$ have real parts localized on the line $\sigma=\frac{1}{2}$. $E_{4n^2}\simeq\frac{\hbar^2}{c^2}\ \omega^2_{2n}$ is the square of the energy of a subtorus of rank one in a maximal pseudo-unramified torus of global level $2n$ where ω_{2n} is the angular frequency of the subtorus.

Remark that the correspondence between Riemann non-trivial zeros and eigenvalues of Hamiltonians has been postulated for a long time (see for example [B-K], [K-S]).

3.2.5 Cuspidal new forms

The non-trivial zeros $\{\lambda_+^{nr}(4n^2,i^2,E_{4n^2})\}_{n=1}$ and $\{\lambda_-^{nr}(4n^2,i^2,E_{4n^2})\}_{n=1}$ of $\zeta_R(s_-)$ and $\zeta_R(s_+)$ allow to consider the following double (pseudo-)unramified simple global elliptic bisemimodule

$$\phi_R^{nr,i}(s_R)\otimes\phi_L^{nr,i}(s_L)=\sum_n(\lambda_-^{nr}(4n^2,i^2,E_{4n^2})\ e^{-2\pi inx}\otimes_D\lambda_+^{nr}(4n^2,i^2,E_{4n^2})\ e^{2\pi inx})$$

constituting, as orbit space, the (pseudo-)unramified supercuspidal representation of the (pseudo-)unramified cuspidal biform $f_R^{nr,\mathrm{new}}(z)\otimes_D f_L^{nr,\mathrm{new}}(z)$ which is a cuspidal "new" biform. Indeed, the new forms $f_R^{nr,\mathrm{new}}(z)$ and $f_L^{nr,\mathrm{new}}(z)$ are defined in the orthogonal complement space with respect to the old forms $f_R^{nr}(z)$ and $f_L^{nr}(z)$ considered above since $f_R^{nr,\mathrm{new}}(z)\otimes_D f_L^{nr,\mathrm{new}}(z)$ constitutes a supercuspidal representation of the Lie algebra $\mathfrak{g}\ell_2(F_{\overline{v}}^{+,T,nr}\times F_v^{+,T,nr})$.

3.2.6 Corollary

The eigenvalues $\lambda_+^{nr}(4n^2,i^2,E_{4n^2})$ and $\lambda_-^{nr}(4n^2,i^2,E_{4n^2})$ for all $n\in\mathbb{N}$ are non-trivial zeros of the Riemann zeta function $\zeta(s)=\sum_n n^{-s}$.

Proof. The set $\{-2n\}$, being trivial zeros of the Riemann zeta functions $\zeta_R(s_-)$ and $\zeta_L(s_+)$, is also a set of trivial zeros of the classical Riemann function $\zeta(s)=\sum_n n^{-s}$.
So, the eigenvalues $\lambda_+^{nr}(4n^2,i^2,E_{4n^2})$ and $\lambda_-^{nr}(4n^2,i^2,E_{4n^2})$, which are localized on the line $\sigma=\frac{1}{2}$ and disposed symmetrically on this line with respect to $\tau=0$ if $s=\sigma+i\tau$, constitute non-trivial zeros of the function $\zeta(s)$. ■

3.3 The Birch and Swinnerton-Dyer conjecture

The conjecture of Birch and Swinnerton-Dyer is closely related and dependent to the conjectures of Shimura-Taniyama-Weil and Riemann. So, we refer to the two first sections of this chapter for the mathematical tools developed there and needed here.

3.3.1. The basic Birch and Swinnerton-Dyer conjecture for an elliptic curve over $\mathbb{Q}$ asserts that the rank of $E(\mathbb{Q})$ is the order of vanishing of its L-function $L(s,E)$ at the point $s=1$ [B-S-D], [Hus], [Rub].

So, we have to study the non-trivial zeros of the L-subseries $L_{R,L}^{nr}(s_\mp,E(\mathbb{Q}))$, i.e. all the zeros of $L_{R,L}^{nr}(s_\mp,E(\mathbb{Q}))$ lying on $\mathrm{Re}(s_\mp)=1$.

3.3.2 The trivial zeros of the restricted zeta functions

In contrast with the Riemann conjecture, the pseudo-unramified L-subseries $L^{nr}_{R,L}(s_\mp, E(\mathbb{Q}))$ are assumed to be restricted in the sense that $L^{nr}_R(s_-, E(\mathbb{Q})) \otimes_D L^{nr}_L(s_+, E(\mathbb{Q})) = \sum\limits_{n_g} \lambda^{nr}_\mp(n_g^2, [m^2])^2 \; n_g^{-2x}$, $n_g \leq n$, $x \in \operatorname{Re}(s_\mp)$, defined over $\mathbb{R}$, is in one-to-one correspondence with the representation of the restricted cuspidal biform $f^{nr}_R(z)_{\rm res} \otimes_D f^{nr}_L(z)_{\rm res}$ degenerating into the restricted simple global elliptic bisemimodule:

$$\phi^{nr,0}_R(s_R)_{\rm res} \otimes_D \phi^{nr,0}_L(s_L)_{\rm res} = \sum_{n_g} (\lambda^{nr}(n_g^2) \; e^{-2\pi i n_g x} \otimes_D \lambda^{nr}(n_g^2) \; e^{2\pi i n_g x}) \; .$$

This global elliptic bisemimodule can be interpreted geometrically as the direct sum of products of irreducible pseudo-unramified semitoric curves corresponding to each other by pairs of same class n_g and constituting the automorphic representation space of the (pseudo-)unramified algebraic bilinear semigroup $\mathrm{GL}_2(F^{+,T,nr}_{\overline{v}_g} \times F^{+,T,nr}_{v_g})$ and the orbit space of $f^{nr}_R(z)_{\rm res} \otimes_D f^{nr}_L(z)_{\rm res}$.

3.3.3 Proposition

The one-to-one correspondence between the double restricted simple global elliptic bisemimodule $2\phi^{nr,0}_R(s_R)_{\rm res} \otimes_D 2\phi^{nr,0}_L(s_L)_{\rm res}$ and the product, right by left, of restricted zeta functions $\zeta^{\rm res}_R(s_-) \otimes_D \zeta^{\rm res}_L(s_+)$ is given by the homomorphism:

$$\begin{aligned} H^{\rm res}_{\phi_{R\times L}\to\zeta_{R\times L}} : \quad & 2\phi^{nr,0}_R(s_R)_{\rm res} \otimes_D 2\phi^{nr,0}_L(s_L)_{\rm res} = \sum_{n_g} (2\lambda^{nr}(n_g^2) \; e^{-2\pi i n_g x} \otimes_D 2\lambda^{nr}(n_g^2) \; e^{2\pi i n_g x}) \\ & \longrightarrow \zeta^{\rm res}_R(s_-) \otimes_D \zeta^{\rm res}_L(s_+) = \sum_{n_g} n_g^{-s_-} \otimes_D n_g^{-s_+}) = \sum_{n_g} n_g^{-2x} \end{aligned}$$

such that its kernel $\operatorname{Ker}(H^{\rm res}_{\phi_{R\times L}\to\zeta_{R\times L}})$ *maps into the set of products, right by left, of trivial zeros of $\zeta^{\rm res}_R(s_-)$ and of $\zeta^{\rm res}_L(s_+)$.*

Proof. This is an adaptation of proposition 3.2.2 to the restricted case. Thus, the kernel $\operatorname{Ker}(H^{\rm res}_{\phi_{R\times L}\to\zeta_{R\times L}})$ is the set of bipoints:

$$\{\sigma_{n_{g_R}} \times \sigma_{n_{g_L}} = 4(\lambda^{nr}(n_g^2))^2 = 2\lambda^{nr}(n_g^2)(e^{-2\pi i n_g x} \mid x=0) \times 2\lambda^{nr}(n_g^2)(e^{2\pi i n_g x} \mid x=0)\}$$

localized on the real axis σ in such a way that the left (resp. right) point $\sigma_{n_{g_L}}$ (resp. $\sigma_{n_{g_R}}$) corresponds to the degeneracy of the irreducible semitoric curve $2\lambda^{nr}(n_g^2) \; e^{2\pi i n_g x})$ (resp. $2\lambda^{nr}(n_g^2) \; e^{-2\pi i n_g x}$).

Remark that $\{\sigma_{n_{g_R}} \times \sigma_{n_{g_L}} = 4(\lambda^{nr}(n_g^2))^2 = 4n_g^2\}_{n_g}$ is the set of trivial zeros of $\zeta^{\rm res}_R(s_-) \otimes_D \zeta^{\rm res}_L(s_+)$. ∎

3.3.4 The non-trivial zeros of the restricted zeta functions

As in section 3.2.3, let $\alpha_{4n_g^2} = \begin{pmatrix} 4n_g^2 & 0 \\ 0 & 1 \end{pmatrix}$ be the split Cartan subgroup element corresponding to the restricted quadratic place $v_{2n_g}^2$.

Let $D_{4n_g^2;i^2} = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ i & 1 \end{smallmatrix}\right)$ be the coset representative of the Lie algebra of the decomposition group acting on $\alpha_{4n_g^2}$.

The infinitesimal generator of the Lie algebra $\mathfrak{gl}_2(F_{\overline{v}_g}^{+,T,nr} \times F_{v_g}^{+,T,nr})$ at the quadratic place $v_{2n_g}^2$ of the Lie group $\mathrm{GL}_2(F_{\overline{v}_g}^{+,T,nr} \times F_{v_g}^{+,T,nr})$ is given by

$$\varepsilon_{4n_g^2} = \begin{pmatrix} E_{4n_g^2} & 0 \\ 0 & 1 \end{pmatrix} .$$

The roots of $\mathfrak{gl}_2(F_{\overline{v}_g}^{+,T,nr} \times F_{v_g}^{+,T,nr})$ are determined by the eigenvalues $\lambda_\pm^{nr}(4n_g^2, i^2, E_{4n_g^2})$ of:

$$D_{4n_g^2} \cdot \varepsilon_{4n_g^2} \cdot \alpha_{4n_g^2} \cdot M_{\frac{1}{2}\to 1} = \left[\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\right] \begin{pmatrix} E_{4n_g^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4n_g^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} ,$$

where the matrix $M_{\frac{1}{2}\to 1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ maps the non-trivial zeros from $\mathrm{Re}(s_+^-) \equiv \sigma = \frac{1}{2}$ to $\sigma = 1$, as it is envisaged in the Birch-Swinnerton-Dyer conjecture. (This map may be of Galois type.) They are given by:

$$\lambda_\pm^{nr}(4n_g^2, i^2, E_{4n_g^2}) = 1 \pm i\sqrt{8n_g^2 E_{4n_g^2} - 1} .$$

3.3.5 Proposition

The roots of $\mathfrak{gl}_2(F_{\overline{v}_g}^{+,T,nr} \times F_{v_g}^{+,T,nr})$ *given by the eigenvalues* $\lambda_\pm^{nr}(4n_g^2, i^2, E_{4n_g^2}) = 1 \pm i\sqrt{8n_g^2 E_{4n_g^2} - 1}$ *of* $D_{4n_g^2;i^2} \cdot \varepsilon_{4n_g^2} \cdot \alpha_{4n_g^2} \cdot M_{\frac{1}{2}\to 1}$ *are the non-trivial zeros of the restricted Riemann zeta functions* $\zeta_R^{\mathrm{res}}(s_-)$ *and* $\zeta_L^{\mathrm{res}}(s_+)$.

Proof. Referring to proposition 3.2.4, we see that the products of the pairs of trivial zeros of the restricted Riemann zeta functions $\zeta_R^{\mathrm{res}}(s_-)$ and $\zeta_L^{\mathrm{res}}(s_+)$ are mapped into the products of the corresponding pairs of the non-trivial zeros according to:

$$\begin{aligned} D_{4n_g^2} \cdot \varepsilon_{4n_g^2} : \quad \det(\alpha_{4n_g^2}) &\longrightarrow \det(D_{4n_g^2} \cdot \varepsilon_{4n_g^2} \cdot \alpha_{4n_g^2} \cdot M_{\frac{1}{2}\to 1})_{\mathrm{ss}} , \\ \{(-2n_g)(-2n_g)\} &\longrightarrow \{\lambda_+^{nr}(4n_g^2, i^2, E_{4n_g^2}) \cdot \lambda_-^{nr}(4n_g^2, i^2, E_{4n_g^2})\} . \end{aligned}$$

Since $D_{4n_g^2} \cdot \varepsilon_{4n_g^2}$ is of Galois type, it maps squares of trivial zeros $(-2n_g)^2$ into products of corresponding pairs of other zeros $\lambda_+^{nr}(4n_g^2, i^2, E_{4n_g^2}) \cdot \lambda_-^{nr}(4n_g^2, i^2, E_{4n_g^2})$ which are non-trivial since their real parts are localized on the real line $\sigma = 1$.
Note that: $E_{4n_g^2N^2} = E_{4n_g^2} \cdot N^2 \simeq \frac{\hbar^2}{c^2}\, \omega^2_{2n_gN}$ is the square of the energy of a quantum (defined in section 2.7 as a $P_2(F^+_{[v_1]})$-subsemimodule of rank N) in a maximal (pseudo-)ramified torus of class $2n_g$ where ω_{2n_gN} is the angular frequency of this quantum.
Remark that $E_{4(n_g-1)^2N^2} \geq E_{4n_g^2N^2}$, which means that the energy of a quantum varies with respect to the (global) level in the sense that when the global level n_g increases, the quantum angular frequency ω tends to decrease (for new arithmetic and algebraic developments of the quanta, see [Pie1]). ■

In the context of the developments of this work, the Birch-Swinnerton-Dyer conjecture can be approached by the following statement:

3.3.6 Proposition

Let $L_{R,L}(s_\mp, E(\mathbb{Q})) \subset L_{R,L}(s_\mp)$ be the L-subseries attached to an elliptic curve over $\mathbb{Q}$. Then, the rank of $E(\mathbb{Q})$ is the order of vanishing of $L^{nr}_{R,L}(s_\mp, E(\mathbb{Q}))$ at $s = 1$ if $L^{nr}_{R,L}(s_\mp, E(\mathbb{Q}))$ is holomorphic at $s = 1$.

Proof:

1. Assume that there is a finite set of P primes p for which the mapping $E(\mathbb{Q}) \to E(\prod\limits_p \mathbb{F}_p)$ is a bijection. Then, to the P Euler factors of $L^{nr}_{R,L}(s_\mp)$ correspond the L-subseries

   $$L^{nr}_{R,L}(s_\mp, E(\mathbb{Q})) = \sum_{n_g} \lambda_\mp^{nr}(n_g^2, m_{n_g}^2)\; n_g^{-s_\mp} \qquad \text{with } N_g \text{ integers “ } n_g \text{ ”} ,$$

   in such a way that N_g corresponds to the number of restricted places of the real semifield $F^+_{R,L}$: the are noted $\overline{v}_g = \{\cdots, \overline{v}_{n_g}, \cdots, \overline{v}_{N_g}\}$ (resp. $v_g = \{\cdots, v_{n_g}, \cdots, v_{N_g}\}$).

2. To $L^{nr}_{R,L}(s_\mp, E(\mathbb{Q}))$ corresponds a restricted simple global elliptic A_{R_g,L_g}-subsemimodule $\phi^{nr,0}_{R,L}(s_{R,L})_{\rm res}$ which is in one-to-one correspondence with the representation space $\mathrm{Rep\ sp}(T_2^t(F^{+,T,nr}_{\overline{v}_g}))$ (resp. $\mathrm{Rep\ sp}(T_2(F^{+,T,nr}_{v_g}))$) of the algebraic semigroup $T_2^t(F^{+,T,nr}_{\overline{v}_g})$ (resp. $T_2(F^{+,T,nr}_{v_g})$) $\subset \mathrm{GL}_2(F^{+,T,nr}_{\overline{v}_g} \times F^{+,T,nr}_{v_g})$. At every place $\overline{v}_{n_g}$ (resp. v_{n_g}) associated with the integer n_g , there is a one-dimensional irreducible curve $E_f(n_g)_{R,L}$, isomorphic to a one-dimensional semitorus $T^1_{R,L}[n_g]$ of class n_g according

to section 3.1. And, thus, at the N_g places, we have the set $\{E_f(n_g)\}_{n_g}$ of N_g irreducible curves of class n_g .

3. Then, a natural surjection exists

$$S_{E_f \to E(\mathbb{Q})} : \quad \{E_f(n_g)\}_{n_g} \longrightarrow E(\mathbb{Q})$$

which identifies the elliptic curve $E(\mathbb{Q})$ with the orbit space of $\{E_f(n_g)\}_{n_g}$ under the action of the semigroup $T_2^t(F_{\overline{v}_g}^{+,T,nr})$ (resp. $T_2(F_{v_g}^{+,T,nr})$) such that $E(\mathbb{Q})$ has N_g generating points.

Then, the rank of $E(\mathbb{Q})$ is $r(E(\mathbb{Q})) = N_g$.

4. Referring to the paper of N. Katz [Kat], the L-series $L_{R,L}(s_\mp)$ are holomorphic at the point $s=1$ as a consequence of a functional equation under $s \to 2-s$. And, thus, if the L-series $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ are also holomorphic at $s=1$, we can speak of the order of vanishing of $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ at $s=1$.

5. Taking into account the proposition 3.3.5 and section 3.3.2, it appears that the pairs of non-trivial zeros of $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ lie on $\mathrm{Re}(s)=1$ and are in one-to-one correspondence with the trivial zeros. And, thus, to each integer n_g of $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ corresponds a trivial zero and a pair of non-trivial zeros.

6. As there are N_g non-trivial zeros of $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$, the order of vanishing of $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ at $s=1$ is N_g . Now, N_g is also the rank of the elliptic curve $E(\mathbb{Q})$ according to 3.

3.3.7 Connecting the Birch-Swinnerton-Dyer conjecture to the Shimura-Taniyama-Weil conjecture

The conjecture of Shimura-Taniyama-Weil, dealing with the connection of the zeros of the L-series $L_{R,L}^{nr}(s_\mp, E(\mathbb{Q}))$ with the rank of the elliptic curve $E(\mathbb{Q})$, seems to take into account only the restricted cuspidal biform $f_R^{nr}(z)_{\rm res} \otimes_D f_L^{nr}(z)_{\rm res}$ degenerating into the restricted simple global elliptic bisemimodule $\phi_R^{nr,0}(s_R)_{\rm res} \otimes_D \phi_L^{nr}(s_L)_{\rm res}$ (see section 3.3.2) in such a way that only one pair of semicircles $T_R^1[n_g]$ and $T_L^1[n_g]$ arises at each class n_g according to proposition 3.3.6.

The general case would refer to the Shimura-Taniyama-Weil conjecture in such a way that the restricted (pseudo-)unramified cuspidal biform of weight two and level 1 ($N=1$)

$$f_R^{nr}(z)_{\rm res} \otimes_D f_L^{nr}(z)_{\rm res} = \sum_{n_g} \left(\lambda_-^{nr}(n_g^2, m_{n_g}^2)\ e^{-2\pi i n_g z} \otimes_D \lambda_+^{nr}(n_g^2, m_{n_g}^2)\ e^{2\pi i n_g z} \right) , \quad z \in \mathbb{C} \ ,$$

be covered by the restricted pseudo-unramified global elliptic bisemimodule:

$$\phi_R^{nr}(s_R)_{\rm res}\otimes_D\phi_L^{nr}(s_L)_{\rm res}=\sum_{n_g}\sum_{m_{n_g}}\left(\lambda_-^{nr}(n_g^2,m_{n_g}^2)\ e^{-2\pi in_gx}\otimes_D\lambda_+^{nr}(n_g^2,m_{n_g}^2)\ e^{2\pi in_gx}\right)\ ,\quad x\in\mathbb{R}$$

(see proposition 3.1.10), in such a way that the modular representation of the elliptic curve $E(\mathbb{Q})$ be given by n_g sets of m_{n_g} products, right by left, of semicircles $T_R^1[n_g,m_{n_g}]$ and $T_L^1[n_g,m_{n_g}]$. Then, the rank $r_{E(\mathbb{Q})}$ of $E(\mathbb{Q})$ would be given by

$$r_{E(\mathbb{Q})}=\bigoplus_{n_g}m_{n_g}\qquad m_{n_g}\geq 1\ .$$

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